Stirling Permutations (Gessel–Stanley 1978)

Def. A permutation of the multiset $\{1,1,\ldots,n,n\}$ is called Stirling if for each $1 \le i \le n$, all #'s between the two occurences of i are greater than i: $\ldots i \underbrace{\qquad}_{\text{all } > i} i \ldots$.

Ex.

1122	1212	2112	2121	2211	1221
✓	×	×	×	✓	✓

Let $Q_n = \{\text{Stirling perm. of size } n\}$.

Prop. $|Q_n| = (2n-1)!!$.

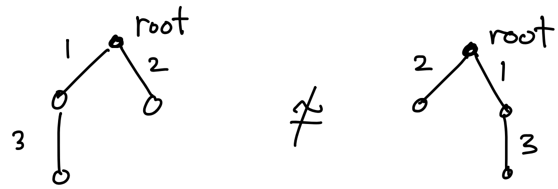
PROOF Where can n's be added to an element of Q_{n-1}? The n's cannot be seperated at all, so they must be added as "nn" to any of the $2(n-1)+1 = 2n-1$ spaces. This sets up a recurrence $|Q_n| = (2n-1)\cdot|Q_{n-1}|$ and the proof follows by induction. $\square$

Name comes from

$$Q_n(t) := \sum_{w \in Q_n} t^{\operatorname{des}(w)} = \frac{(1-t)^{2n+1}}{t} \sum_{m \ge 0} \underbrace{S(m+n,m)}_{\text{(Stirling \# of second kind)}} t^m .$$

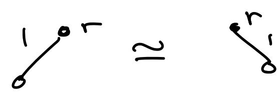
Trees

Def./Ex. : Rooted labeled plane tree of size n

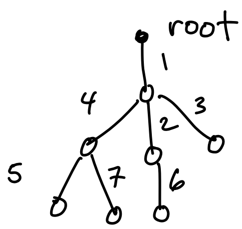


(size 3)

(left-right matters when you have multiple children)

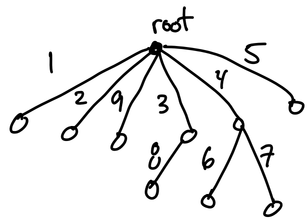


Depth first search choosing left :



$\longrightarrow$ $14557742662331 \in Q_7$.

$11229938834667745 5 \in Q_8$ $\longrightarrow$



Prop. $Q_n \longleftrightarrow \{\text{rooted increasing plane trees of size } n\}$

ϕ $\quad\quad\quad\quad$ $\text{!!} \;\; RIP(n)$

Flattened Stirling:

Given $w \in Q_n$, break into its ascending runs, e.g.

$$\underbrace{12333}_{1}\,\underbrace{2}_{2}\,\underbrace{1445566778899}_{1} \in Q_9 \setminus \overline{Q}_9$$

1 2 1 ← leading terms of runs.

Def. $w \in Q_n$ is flattened if leading terms of runs are weakly increasing.

Set $\overline{Q}_n = \{ w \in Q_n : w \text{ is flattened} \} \subseteq Q_n$.

Ex.

1122	2211	1221
✓	×	✓

Prop. $|\overline{Q}_n| = (n-1)^{st}$ Dowling # D_{n-1}.

PROOF Bijection to type B set partitions.

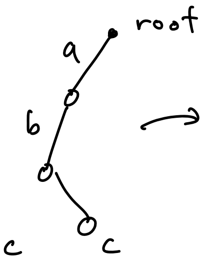
(see Buck et al. 2023) □

Research Question ①: Recall $\phi : Q_n \to RIP(n)$ is a bijection.

Identify $\phi(\bar{Q}_n) \subseteq RIP(n)$.

Ideas:

① "Depth" ≤ 2 :



$\longrightarrow$ $a \ldots b \ldots c \ldots c \ldots b \ldots a$

$a < b < c$

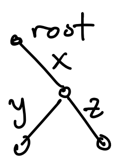
$\underbrace{abc}\underbrace{cba}$ not flat!

depth 3

$\Rightarrow$ these trees are wide & not tall.

② Grandchildren of the root (depth 2) must be increasing from left to right:

$x < y$
$x < z$



$\longrightarrow$ $\cdots x \cdots y \cdots y \cdots z \cdots z \cdots x \cdots$

if $y > z$: $\underbrace{xyy}\underbrace{zz}\underbrace{x}$ not flat!

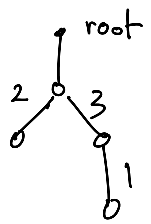
$\Rightarrow$ ① & ② are necessary for $\phi(\bar{Q}_n)$.

Are they sufficient?

Research Question (2): Define & study flattened quasi-Stirling permutations.

Def. (2018) quasi-Stirling perms. are the perms. of $\{1,1,\ldots,n,n\}$ that map to $RP(n)$ (removed increasing requirement)

Prop. # quasi-Stirling of size n is $n! \cdot$ (nth Catalan #).



$\longrightarrow$ $\underbrace{223}\underbrace{113}$

Not flat

Def.? quasi-Stirling flattened if leading terms of runs in weakly increasing order

Probably not a good definition because enumeration starts as

1, 2, 7, 37, 250, 2006, . . . ← not in OEIS.

primes :(