Homework 1
Spring 2025

These exercises are from our textbook for the course:
Graphs & Digraphs, Seventh Edition.

1. A graph G has order $n = 3k + 3$ for some positive integer k. Every vertex of G has degree $k+1$, $k+2$ or $k+3$. Prove that G has at least $k+3$ vertices of degree $k+1$ **or** at least $k+1$ vertices of degree $k+2$ **or** at least $k+2$ vertices of degree $k+3$.

Solution: Let a, b and c denote the number of vertices in G of degree $k+1$, $k+2$ and $k+3$, respectively. For the sake of contradiction, suppose $a \leq k+2$ and $b \leq k$ and $c \leq k+1$. Since $a+b+c = 3k+3$, it follows we must have $a = k+2$ and $b = k$ and $c = k+1$. By the handshaking lemma,

$$\sum_{v \in V(G)} \deg_G(v) = (k+2)(k+1) + k(k+2) + (k+1)(k+3) = 3k^2 + 9k + 5$$

must be an even integer. However, if k is even then $3k^2 + 9k$ is also even so then $3k^2 + 9k + 5$ is odd. Or if k is odd then $3k^2$ and $9k$ are both odd so that $3k^2 + 9k$ is even and thus $3k^2 + 9k + 5$ is odd. In any case, we have reached a contradiction.

2. Prove that every bipartite graph of order n has at most $\lfloor n^2/4 \rfloor$ edges, and that there is exactly one such graph up to isomorphism.

Solution: Let G be a bipartite graph of order n with $V(G) = A \sqcup B$, $a = |A|$, $b = |B|$. The size of G is at most equal to the size of the complete bipartite graph $K_{a,b}$ which, by definition, can be obtained from G by adding any remaining edges between A and B. The size of $K_{a,b}$ is equal to

$$\underbrace{b + b + \ldots + b}_{a \text{ copies}} = ab = a(n-a) \leq \frac{[a + (n-a)]^2}{4} = \frac{n^2}{4}, \qquad (\star)$$

where the inequality follows from the AM-GM inequality. Since the size of $K_{a,b}$ is an integer this implies $|E(G)| \leq |E(K_{a,b})| \leq \lfloor n^2/4 \rfloor$. Equality in $(\star)$ only occurs when n is even, in which case $a = b = n/2$ and $G \simeq K_{n/2,n/2}$. Otherwise, if n is odd and G has size $\lfloor n^2/4 \rfloor$ then, using the fact that $ab = \lfloor n^2/4 \rfloor = \lfloor n/2 \rfloor \lceil n/2 \rceil$, we have $G \simeq K_{a,b} \simeq K_{b,a}$, where the existence of the last isomorphism is evident.

3. Let G_1 and G_2 be self-complementary graphs, and G_2 have even order n. Let G be the graph obtained from G_1 and G_2 by joining each vertex of G_2 whose degree is less than $n/2$ to every vertex of G_1. Show that G is self-complementary.

Solution: By assumption, there are graph isomorphisms

$$\varphi_1 : V(G_1) \to V(\overline{G_1}) \quad \text{and} \quad \varphi_2 : V(G_2) \to V(\overline{G_2}).$$

With G formed as prescribed, define a map $\varphi : V(G) \to V(\overline{G})$ by

$$\varphi(v) = \begin{cases} \varphi_1(v) & v \in V(G_1); \\ \varphi_2(v) & v \in V(G_2). \end{cases}$$

We claim that φ is a graph isomorphism, hence we need to show $u \sim v$ in G if and only if $\varphi(u) \sim \varphi(v)$ in $\overline{G}$.

($\Rightarrow$) Suppose $u \sim v$ for some $u, v \in V(G) = V(G_1) \sqcup V(G_2)$.

Case 1: If both $u, v \in V(G_1)$ then we know $\varphi(u) = \varphi_1(u) \sim \varphi_1(v) = \varphi(v)$ in $\overline{G}$ since φ_1 is a graph isomorphism.

Case 2: If both $u, v \in V(G_2)$ then we know $\varphi(u) = \varphi_2(u) \sim \varphi_2(v) = \varphi(v)$ in $\overline{G}$ since φ_2 is a graph isomorphism.

Case 3: If $u \in V(G_1)$ and $v \in V(G_2)$ then, since $u \sim v$ in G, we know that $\deg_{G_2}(v) < n/2$. Since graph isomorphisms preserve vertex degree, we have that $\deg_{\overline{G_2}}(\varphi(v)) = \deg_{G_2}(v)$ which implies $\deg_{\overline{G_2}}(\varphi(v)) < n/2$. We also know that $\deg_{G_2}(\varphi(v)) + \deg_{\overline{G_2}}(\varphi(v)) = n - 1$. Hence

$$\deg_{G_2}(\varphi(v)) = n - 1 - \deg_{\overline{G_2}}(\varphi(v)) > \frac{n}{2} - 1,$$

which implies $\deg_{G_2}(\varphi(v)) \geq n/2$. In particular, this means $\varphi(u) \nsim \varphi(v)$ in G, which implies $\varphi(u) \sim \varphi(v)$ in $\overline{G}$ as desired.

($\Leftarrow$) We prove the contrapositive. Suppose $u \nsim v$ in G. Then, using the same tools from before we see $\deg_{\overline{G_2}}(\varphi(v)) = \deg_{G_2}(v) \geq n/2$ which implies

$$\deg_{G_2} \varphi(v) = n - 1 - \deg_{\overline{G_2}} \varphi(v) \leq \frac{n}{2} - 1.$$

Hence $\deg_{G_2}(\varphi(v)) < n/2$ which implies $\varphi(u) \sim \varphi(v)$ in G and thus we have $\varphi(u) \nsim \varphi(v)$ in $\overline{G}$, as desired.

This establishes that G and $\overline{G}$ are isomorphic as graphs, so G is self-complentary.

4. Prove that every non-trivial circuit in a graph contains a cycle.

Solution: Let $C : v = v_0, v_1, \ldots, v_n = v$ be a non-trivial circuit. If the vertices v_i are distinct for all $0 \leq i < n$ then C is a cycle. Otherwise, consider the set

$$\{(k, \ell) : k < \ell \text{ and } v_k = v_\ell\}$$

and choose a pair (k, ℓ) that minimizes $\ell - k$. This minimality ensures there are no $k < s, t < \ell$ for which $v_s = v_t$ or $v_k = v_t$, hence there are no repeated vertices in the subcircuit $C' = v_k, \ldots v_\ell$, thus C' is a cycle.