Homework 2
Spring 2025

These exercises are from our textbook for the course:
Graphs & Digraphs, Seventh Edition.

1. For which integers x with $0 \leq x \leq 7$ is the sequence $(7, 6, 5, 4, 3, 2, 1, x)$ graphical?

Solution: To start, the sum of the degrees needs to be even and so it follows that x must be even. We check each case using the Havel–Hakimi theorem. In particular, it follows from the theorem that for $s = (d_1 \geq \ldots \geq d_n)$ to be graphical it is necessary for d_1 to be strictly less than the number of non-zero d_i in s, otherwise s' will contain a -1 and not be graphical. We have the cases:

> $x = 0$: We get $(7, 6, 5, 4, 3, 2, 1, 0)$ is not graphical by our observation above. So $x = 0$ doesn't work.
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> $x = 2$: We get $(7, 6, 5, 4, 3, 2, 2, 1)$ is graphical if and only if $(5, 4, 3, 2, 1, 1, 0)$ is graphical, which is graphical if and only if $(3, 2, 1, 0, 0, 0)$ is graphical, but it's not by our observation above. So $x = 2$ doesn't work.
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> $x = 4$: We get $(7, 6, 5, 4, 4, 3, 2, 1)$ is graphical if and only if $(5, 4, 3, 3, 2, 1, 0)$ is graphical, which is graphical if and only if $(3, 2, 2, 1, 0, 0)$ is graphical, which is graphical if and only if $(1, 1, 0, 0, 0)$ is graphical, which is the degree sequence of the graph $G = (\{v_1, \ldots, v_5\}, \{v_1v_2\})$. So $x = 4$ works.
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> $x = 6$: We get $(7, 6, 6, 5, 4, 3, 2, 1)$ is graphical if and only if $(5, 5, 4, 3, 2, 1, 0)$ is graphical, which is graphical if and only if $(4, 3, 2, 1, 0)$ is graphical, but it's not by our observation above. So $x = 6$ doesn't work.

We conclude that only $x = 4$ makes the given sequence graphical.

2. Characterize those graphs with the property that every connected subgraph is an induced subgraph.

Solution: Let G be a graph. We will show that

$$\text{every connected subgraph of } G \text{ is an induced subgraph} \iff G \text{ is a forest.}$$

($\Rightarrow$) If G contains a cycle $C = u_1 \ldots u_k u_1$ then the path $P = u_1 \ldots u_k$ is a connected subgraph that is not an induced subgraph. Hence it is necessary for G to be acyclic, i.e., for G to be a forest.

($\Leftarrow$) Let H be a connected subgraph of G. To show H is induced, let $u, v \in V(H)$ such that $uv \in E(G)$. As G is a forest we know the edge $uv \in E(G)$ is the unique path from u to v in G. Hence we must have the edge $uv \in E(H)$ since H is connected, thus we conclude that H is induced.

3. Prove that every tree with maximal degree k has at least k leaves.

Solution: A tree with maximal degree two is isomorphic to a path, which has two leaves. It follows that any tree with at least two vertices has at least two leaves. Now suppose T is a tree with maximal degree k and let v be a vertex of degree k in T. The graph $T - v$ is a forest with k components $T_1, \ldots, T_k$ that are each trees. Any T_i with order one is a leaf in T, and each T_i with order at least two has at least two leaves in T_i. Perhaps one of these leaves of T_i are adjacent to v in T, but they cannot both be adjacent to v otherwise T would contain a cycle. Hence each T_i for $1 \le i \le k$ contributes at least one leaf to T and so T has at least k leaves.

4. Let $s = (d_1 \ge \ldots \ge d_n)$ be a sequence of non-negative integers and set $p = \sum_{i=1}^{n} d_i$. Prove that s is the degree sequence of a multigraph if and only if p is even and $d_1 \le p/2$.

Solution: ($\Rightarrow$) Suppose G is a multigraph with degree sequence s. We know $\sum d_i = p$ must be even by the Handshaking Lemma. Of course $d_1 \le |E(G)|$, which implies $2d_1 \le 2|E(G)| = p$ again by the Lemma and thus $d_1 \le p/2$.
($\Leftarrow$) We will construct a multigraph with degree sequence s by induction on the non-negative even integers p. If $p = 0$ then the empty graph of order n is a multigraph with degree sequence s. Now let $p \ge 2$ and note this implies $d_1 \ge 1$. For our inductive hypothesis we assume there is a multigraph for any non-increasing sequence of non-negative integers that sum to any even less than p. Since $d_1 \le p/2$ it follows that $d_1 \le d_2 + \ldots + d_n$. If we have equality here then the graph obtained by adding d_i edges between vertices v_1 and v_i for $2 \le i \le n$ is a multigraph with degree sequence s. Otherwise, since $1 \le d_1 < d_2 + \ldots + d_n$ it follows there are at least two non-zero d_i. Let s' be a new sequence of length n obtained from s by decreasing the two smallest non-zero entries of s by one. Then s' is a non-increasing sequence of non-negative integers that sums to $p - 2$, hence by our inductive hypothesis there is a multigraph G' of order n with degree sequence s'. Now let G be the graph obtained from G' by adding a single edge between the vertices corresponding to the d_i we decremented to get s' from s. The graph G is a multigraph with degree sequence s as desired.