Homework 3
Spring 2025

These exercises are from our textbook for the course:
Graphs & Digraphs, Seventh Edition.

1. Let D be a connected digraph.

   a.) Show D is Eulerian if and only if $\mathrm{od}(v) = \mathrm{id}(v)$ for every vertex v in D.

   Solution: ($\Rightarrow$) If $C : v_1 \dots v_n v_1$ is an Eulerian cycle then each occurrence of a vertex v_i in C contributes exactly one to the in-degree of v_i and one to the out-degree of v_i because edges are not repeated in C. The result follows.
   ($\Leftarrow$) If $\mathrm{od}(v) = \mathrm{id}(v)$ then $\deg(v) = \mathrm{od}(v) + \mathrm{id}(v)$ must be even. Hence if $\mathrm{od}(v) = \mathrm{id}(v)$ for every vertex v in D then every vertex of D has even degree, which we know is equivalent to D being Eulerian.

   b.) Show D has an Eulerian trail if and only if it contains vertices u, w with

   $$\mathrm{od}(u) = \mathrm{id}(u) + 1, \quad \mathrm{od}(w) = \mathrm{id}(w) - 1$$

   and $\mathrm{od}(v) = \mathrm{id}(v)$ for every other vertex v in D. Moreover, every Eulerian trail in D starts at u and ends at w.

   Solution: ($\Rightarrow$) Suppose $v_1 \dots v_n$ is an Eulerian trail in D. Consider the graph D' that adds the arc (v_n, v_1) to D. Then D' is Eulerian and so by 1(a) we know $\mathrm{od}(v) = \mathrm{id}(v)$ for each vertex v in D'. So then D, with the arc (v_n, v_1) removed, has the desired property with $u = v_1$ and $w = v_n$.
   ($\Leftarrow$) If we have such a digraph D then D' obtained by adding the arc (w, u) to D has the property that $\mathrm{id}(v) = \mathrm{od}(v)$ for each vertex in D'. Therefore D' has an Eulerian circuit $C : u \dots wu$ and thus $P : u \dots w$ is an Eulerian trail in D.

   For the second part, let $v_1 \dots v_n$ be an Eulerian trail in D. Then we know $u = v_i$ for some $1 \leq i \leq n$. From our argument in 1(a) we know $\mathrm{od}(v_k) = \mathrm{id}(v_k)$ for each $1 < k < n$. Moreover, we know $\mathrm{od}(v_1) = \mathrm{id}(v_1) + 1$ because of this first appearance of v_1. Hence we must have that $u = v_1$. Similarly, we know that $\mathrm{od}(v_n) = \mathrm{id}(v_n) - 1$ because of this last appearance of v_n. Therefore we must have that $w = v_n$ and so every Eulerian trail in D starts at u and ends at w.

2. Let D be a connected digraph such that

$$\sum_{v \in V(D)} |\mathrm{od}(v) - \mathrm{id}(v)| = 2t$$

for t a positive integer. Prove that $E(D)$ can be partitioned into subsets $E_1, \ldots, E_t$ so that the subgraph $G[E_i]$ induced by E_i is an open trail for every i.

Solution: We use induction on $t \geq 1$. Define the *discrepancy* of a digraph to be

$$\mathrm{disc}(D) = \sum_{v \in V(D)} |\mathrm{od}(v) - \mathrm{id}(v)|.$$

Suppose D is connected and $\mathrm{disc}(D) = 2$. As $\mathrm{disc}(D)$ is a sum of non-negative integers, this implies either $\mathrm{disc}(D) = 2 + 0 + \ldots$ or $\mathrm{disc}(D) = 1 + 1 + 0 + \ldots$, but the former is not possible since $\sum \mathrm{od}(v) = \sum \mathrm{id}(v)$. And for the latter, that same fact implies we must have two vertices, say u and w, with $\mathrm{od}(u) = \mathrm{id}(u) + 1$ and $\mathrm{od}(w) = \mathrm{id}(w) - 1$. Hence by Exercise 1(b) we know that D contains an Eulerian trail. So $E_1 = E(D)$ works since $G[E_1] = D$ is an open trail as desired.

Now assume that any connected digraph E with $0 < \mathrm{disc}(E) < 2t$ has the desired property. Let D be a connected digraph with $\mathrm{disc}(D) = 2t$. Then there must be some vertices in D that have a greater out-degree than in-degree, and some with greater in-degree than out-degree. Pick a vertex v of the former type. We can construct a path P to a vertex w of the latter type by repeatedly visiting neighbors. Initialize E_1 to be $E(P)$. Consider $D' = D - E(P)$. We have $\mathrm{disc}(D') = 2t - 2$ since the out-degree of v and the in-degree of w both decreased by one, and the remaining vertices in the path lost one to both their in- and out-degrees. However, the digraph D' may not be connected, so we cannot apply our inductive hypothesis to D' directly. Consider the connected components of D'. Add the edges of any Eulerian connected component of D' (i.e., any connected component with a discrepancy of zero) to E_1. We can apply the inductive hypothesis to each of the remaining connected components of D' to partition their edges and overall get the remaining $E_2, \ldots, E_t$ as desired.

3. Show for G and H Hamiltonian graphs that $G \square H$ is also Hamiltonian. What does this say about the hypercube Q_n?

> **Solution:** Suppose G has order m and H has order n. Since G is Hamiltonian, it contains a subgraph isomorphic to C_m. Likewise, H contains a subgraph isomorphic to C_n. It follows that $G \square H$ contains a subgraph isomorphic to $C_m \square C_n$, and this subgraph contains all the vertices of $G \square H$. Hence it suffices to show that $C_m \square C_n$ contains a Hamiltonian cycle. To do this, write $V(C_m) = \{u_1, \ldots, u_m\}$ and $V(C_n) = \{v_1, \ldots, v_n\}$ and arrange the vertices of $C_m \square C_n$ into a $m \times n$ grid so that vertex labels increase left-to-right down rows and top-to-bottom down columns. Start in the top left corner (u_1, v_1) and move down the first row to (u_1, v_n). Now we consider two cases based on the parity of m.
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> If m is even, travel down one to (u_2, v_n) and then move left along the row to (u_2, v_1). If $m = 2$ then move to (u_1, v_1) and we're done. Else, move down to (u_3, v_1) and repeat the process again to end up at (u_4, v_1), etc. Continue in this fashion to reach (u_6, v_1) and so forth until you eventually reach (u_m, v_1), then move back to (u_1, v_1) to complete a Hamiltonian cycle.
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> If m is odd, travel all the way down to (u_m, v_n). From here, move left once to (u_m, v_{n-1}) and then all the way up to (u_2, v_{n-1}). Then move left once to (u_2, v_{n-2}) and then all the way down to (u_m, v_{n-2}). Continue doing this. If n is even we will eventually reach (u_2, v_1). If n is odd we will eventually reach (u_m, v_1). In either case, we can now move to (u_1, v_1) to complete the Hamiltonian cycle.

4. Prove the graph $K_{3r,2r,r}$ is Hamiltonian for every positive integer r but that $K_{3r+1,2r,r}$ is **not** Hamiltonian for any positive integer r.

> **Solution:** Consider $G = K_{3r,2r,r}$ for some $r \geq 1$. Write $V(G) = A_1 \sqcup A_2 \sqcup A_3$ where $|A_i| = ir$. Since each vertex of A_k is adjacent to every vertex of A_ℓ for $k \neq \ell$, we have $\deg(v) = (6 - i)r$ for each $v \in A_i$. In particular, $\deg(v) \geq 3r = |V(G)|/2$ for all $v \in V(G)$ and therefore G is Hamiltonian by Dirac's theorem.
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> Now consider $H = K_{3r+1,2r,r}$ for some $r \geq 1$. Write $V(H) = A_1 \sqcup A_2 \sqcup A_3$ as before, except now $|A_3| = 3r + 1$ and the degree of the vertices in A_1 and A_2 are each increased by one. Observe that for $S = A_1 \sqcup A_2$, the graph $H - S = H[A_3]$ is isomorphic to the empty graph of order $3r + 1$. Hence the number of connected components of $H - S$ is greater than $3r = |S|$, which implies H is not Hamiltonian.