Quiz 10

Due: 12 November 2024

Answer the questions in the spaces provided. **Show all of your work and circle the answer you would like to have graded for each question.**

Name: ______________________________

1. Find the exact value of $\cos(105°)$ by hand.

Solution: Since $105° = 60° + 45°$, by the sum formula for cosine we have

$$\begin{aligned}\cos(105°) &= \cos(60°)\cos(45°) - \sin(60°)\sin(45°)\\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \boxed{\frac{\sqrt{2}}{4}(1-\sqrt{3}).}\end{aligned}$$

2. Find all solutions to the equation $\sin(2x)\cos(5x) + \sin(5x)\cos(2x) = 0$.

Solution: The sum formula for sine says $\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) = \sin(\alpha+\beta)$. Applying this to $\alpha = 2x$ and $\beta = 5x$ turns our equation into

$$0 = \sin(2x)\cos(5x) + \sin(5x)\cos(2x) = \sin(2x+5x) = \sin(7x).$$

Notice that $\sin(7x) = 0$ whenever $7x = \pi n$ for some integer n. Hence the solutions to the given equation are of the form $\boxed{x = \frac{\pi}{7}n.}$

3. Find the exact value of the product $\cos\left(\frac{11\pi}{12}\right) \cdot \cos\left(\frac{\pi}{12}\right)$ by hand.

Solution: The product-to-sum formula tells us that

$$\begin{aligned}\cos\left(\frac{11\pi}{12}\right) \cdot \cos\left(\frac{\pi}{12}\right) &= \frac{1}{2}\left[\cos\left(\frac{11\pi}{12} - \frac{\pi}{12}\right) + \cos\left(\frac{11\pi}{12} + \frac{\pi}{12}\right)\right] \\ &= \frac{1}{2}\left[\cos\left(\frac{5\pi}{6}\right) + \cos(\pi)\right] \\ &= \boxed{-\frac{1}{2}\left(\frac{\sqrt{3}}{2} + 1\right)}.\end{aligned}$$

4. Verify the identity

$$\frac{\sin(4x) + \sin(2x)}{\cos(4x) - \cos(2x)} = \cot(x).$$

Hint: Use sum-to-product formulas.

Solution: We use the sum-to-product formulas for both sine and cosine to compute

$$\frac{\sin(4x) + \sin(2x)}{\cos(4x) - \cos(2x)} = \frac{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)}{-2\sin\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right)} = -\frac{\cos(\frac{2x}{2})}{\sin(\frac{2x}{2})} = -\frac{\cos(x)}{\sin(x)} = -\cot(x).$$

So the given identity is not true, for example, at $x = \pi/4$. I think I meant to write $-\cot(x)$ on the right hand side.

5. Consult the picture and solve for the exact value of α:
Hint: Observe that $\tan(\alpha + \beta) = 10$.

Solution: The sum formula for tangent gives

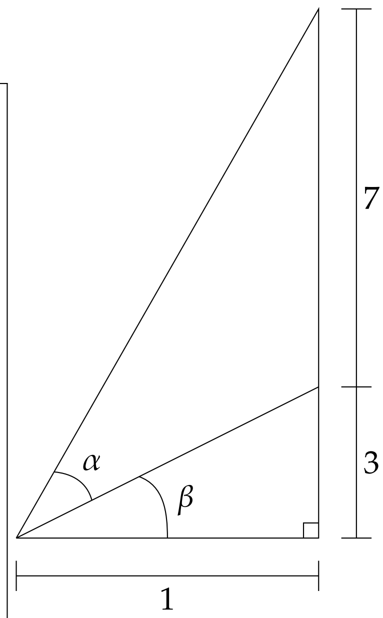
$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha) \cdot \tan(\beta)}. \qquad (\star)$$

By consulting the picture we determine $\tan(\beta) = 3$ and $\tan(\alpha + \beta) = 10$. Plugging this into $(\star)$ yields

$$10 = \frac{\tan(\alpha) + 3}{1 - 3\tan(\alpha)}.$$

This is equivalent to $10 - 30\tan(\alpha) = \tan(\alpha) + 3$, or $7 = 31\tan(\alpha)$. Hence $\tan(\alpha) = 7/31$ which implies

$$\boxed{\alpha = \arctan(7/31) \approx 12.72^\circ.}$$



(Not drawn to scale.)