Quiz 3
11 June 2024

Answer the questions in the spaces provided. **Show all of your work and circle the answer you would like to have graded for each question.**

Name: ______________________________

1. Compute **all four second-order** partial derivatives of $f(x, y) = e^{xy+y}$.

Solution: We use the rules of partial differentiation to first compute that

$$f_x(x, y) = ye^{xy+y} \quad \text{and} \quad f_y(x, y) = (x + 1)e^{xy+y}.$$

We use these to further compute

$$\begin{aligned} f_{xx}(x, y) &= y^2 e^{xy+y} \\ f_{xy}(x, y) &= e^{xy+y} + (x + 1)ye^{xy+y} \\ f_{yx}(x, y) &= e^{xy+y} + (x + 1)ye^{xy+y} \\ f_{yy}(x, y) &= (x + 1)^2 e^{xy+y}. \end{aligned}$$

2. Argue why the function $f(x, y) = \dfrac{3x^2 y}{x^4 + y^2}$ is **not differentiable** at $(0, 0)$.

Solution: We claim that $f(x, y)$ is **not continuous** at $(0, 0)$ and therefore cannot be differentiable at $(0, 0)$. To see this, observe what happens in the limit as $f(x, y)$ approaches $(0, 0)$ along the y-axis where $x = 0$:

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(0,y)\to(0,0)} \frac{3 \cdot 0^2 \cdot y}{0^4 + y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

However, in the limit as $f(x, y)$ approaches $(0, 0)$ along the curve $y = x^2$ we see

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,x^2)\to(0,0)} \frac{3x^2 \cdot x^2}{x^4 + x^4} = \lim_{x\to 0} \frac{3x^4}{2x^4} = \frac{3}{2}.$$

Since $0 \neq \frac{3}{2}$ we conclude that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, proving our claim.

3. Find the linear approximation of $f(x, y) = \sqrt{xy}$ at the point $(1, 4)$ and then **use it to estimate** $f(0.98, 4.04)$.

Solution: In general, the formula for we're looking at here is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

so that $L(x, y) \approx f(x, y)$ for points (x, y) that are near to the point (a, b). The point $(1, 4)$ will serve as our (a, b). Compute $f(a, b) = f(1, 4) = \sqrt{4} = 2$. Further,

$$f_x(x, y) = \frac{y}{2\sqrt{xy}} \quad \text{and} \quad f_y(x, y) = \frac{x}{2\sqrt{xy}}$$

so that $f_x(a, b) = f_x(1, 4) = 1$ and $f_y(a, b) = f_y(1, 4) = 1/4$. Hence

$$f(x, y) \approx 2 + (x - 1) + \frac{1}{4}(y - 4)$$

for points (x, y) that are near to the point $(1, 4)$.

From this we estimate

$$\begin{aligned} f(0.98, 4.04) &\approx 2 + (0.98 - 1) + \frac{1}{4}(4.04 - 4) \\ &= 2 - 0.02 + 0.01 \\ &= 1.99. \end{aligned}$$