Quiz 8

6 August 2024

> Answer the questions in the spaces provided. **Show all of your work and circle the answer you would like to have graded for each question.**

Name: ______________________________

1. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and cylinder $x^2 + y^2 = 1$ oriented counter-clockwise from above.

Solution: The curve C bounds an elliptical region S in the plane $y + z = 2$ and we can apply Stokes' theorem. First compute

$$\begin{aligned}\operatorname{curl} \vec{F} &= \left\langle \frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(x), \frac{\partial}{\partial z}(-y^2) - \frac{\partial}{\partial x}(z^2), \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y^2) \right\rangle \\ &= \langle 0 - 0, 0 - 0, 1 - (-2y) \rangle \\ &= \langle 0, 0, 1 + 2y \rangle.\end{aligned}$$

By orienting S upward, its boundary curve C has positive orientation. Hence, using that $z = 2 - y = g(x, y)$, we parameterize S via $\vec{r}(x, y) = \langle x, y, 2 - y \rangle$ with domain D given by the disk $x^2 + y^2 \leq 1$. The normal vector to S here is $\langle -g_x, -g_y, 1 \rangle = \langle 0, 1, 1 \rangle$. Thus by Stokes' theorem we compute

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} &= \iint_D \langle 0, 0, 1 + 2r \sin\theta \rangle \cdot \langle 0, 1, 1 \rangle \, dA \\ &= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta) r \, dr \, d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3} \sin\theta \right) d\theta \\ &= (\pi - 2/3) - (0 - 2/3) \\ &= \boxed{\pi.}\end{aligned}$$

2. Evaluate $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle xz, yz, xy \rangle$ and S is the upwardly oriented part of the sphere $x^2 + y^2 + z^2 = 4$ inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

Solution: The curve of intersection between the sphere and the cylinder is a boundary curve for the surface S and we can apply Stokes' theorem. Set the surface equations equal to each other to find that the intersect in a unit circle on the plane $z = \sqrt{3}$. Since S is oriented upward, we want $C = \partial S$ to travel counter-clockwise from above. We parameterize C by $\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$ for $0 \le t \le 2\pi$. Hence by Stokes' theorem we have that

$$\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt.$$

To compute the line integral in the last equation we notice that

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle = 0.$$

Thus the corresponding line integral is zero and so overall we have shown

$$\boxed{\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = 0.}$$