Warm-up — 28 October 2021

1. Write the region

$$R = \{(x, y) : 0 \leq x \leq \sqrt{6y - y^2} \text{ and } 0 \leq y \leq 6\}$$

in polar coordinates.

Solution: Sketch the region R in the xy-plane and observe that R is the right half of a disk with radius 3 centered at $(0, 3)$. As R is entirely contained in the first quadrant it follows that $0 \leq \theta \leq \frac{\pi}{2}$. Moreover, the equation of the circle that encloses R is $x^2 + (y - 3)^2 = 9$, which implies $r^2 - 6r \sin\theta = 0$ and thus we see that $0 \leq r \leq 6 \sin\theta$. Hence we have

$$\begin{aligned} R &= \{(x, y) : 0 \leq x \leq \sqrt{6y - y^2} \text{ and } 0 \leq y \leq 6\} \\ &= \boxed{\{(r, \theta) : 0 \leq r \leq 6 \sin\theta \text{ and } 0 \leq \theta \leq \pi/2\}.} \end{aligned}$$

2. Convert the following integral to polar coordinates:

$$\int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} \frac{1}{x^2 + y^2} \, dy \, dx.$$

Solution: Sketch the region of integration in the xy-plane to find that we're integrating over the portion of the disk with radius two centered at the origin that lies in the second quadrant. Hence this region can be described by the points (r, θ) such that $0 \leq r \leq 2$ and $\frac{\pi}{2} \leq \theta \leq \pi$ and therefore, by a change of variables to polar coordinates, the integral becomes

$$\int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} \frac{1}{x^2 + y^2} \, dy \, dx = \boxed{\int_{\frac{\pi}{2}}^{\pi} \int_{0}^{2} \frac{1}{r^2} \, r \, dr \, d\theta.}$$

3. Find the volume of the solid bounded by the plane $z = -1$ and the elliptic paraboloid $z = 3 - x^2 - y^2$.

Solution: First, observe that these surfaces intersect when $-1 = 3 - x^2 - y^2$ which is the circle $x^2 + y^2 = 4$ of radius two. So the solid E that we're interested in lies below the paraboloid $z = 3 - x^2 - y^2$ and above the circle of radius two about the z-axis resting on the plane $z = -1$. To carry out the integration we first convert to polar coordinates. From our observations above it follows that the region of integration is $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$, and thus

$$\begin{aligned}
\text{the volume of } E &= \iint_E [3 - x^2 - y^2 - (-1)] \, dV \\
&= \int_0^{2\pi} \int_0^2 (4 - r^2) \, r \, dr \, d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^2 (4r - r^3) \, dr \\
&= 2\pi \left[2r^2 - \frac{r^4}{4} \right]_{r=0}^{r=2} \\
&= \boxed{8\pi.}
\end{aligned}$$