**Warm-up** — 16 September 2021

1. Find an equation for the surface $S$ that consists of all points equidistant from the point $(2,0,0)$ and the plane $x=-2$. Identify the traces parallel to the $yz$-plane and classify the surface. What can you say about the traces parallel to the $xy$-plane? What about the $xz$-plane?

**Solution:** Pick an arbitrary point $(x,y,z)$ that lies on the given surface $S$. The distance from $(x,y,z)$ to the plane $x=-2$ is $\sqrt{(x+2)^2}$. Similarly, the distance from $(x,y,z)$ to the point $(2,0,0)$ is $\sqrt{(x-2)^2+y^2+z^2}$. Since $(x,y,z)$ lies on $S$ we must have that these two distances are equal, therefore

$$\sqrt{(x+2)^2}=\sqrt{(x-2)^2+y^2+z^2}$$

and now by squaring this equation and expanding the product we obtain

$$x^2+4x+4=x^2-4x+4+y^2+z^2.$$

Simplifying the above expression yields $8x=y^2+z^2$ which is equivalent to

$$x=\frac{y^2}{8}+\frac{z^2}{8}.$$

This shows that $S$ is a circular paraboloid: the traces of planes $x=k$ parallel to the $yz$-plane are circles of radius $\sqrt{8k}$; the traces of planes $y=k$ parallel to the $xz$-plane are parabolas and same goes for planes parallel to the $xy$-plane.

2. Do the curves $\mathbf{r}_1(t)=\langle t,3-t,9+t^2\rangle$ and $\mathbf{r}_2(s)=\langle 1-s,s+2,s^2\rangle$ intersect? If so, determine the point of intersection and the value of $\cos\theta$ where $\theta$ is their (acute) angle of intersection.

**Solution:** To check for a point of intersection, set $\mathbf{r}_1(t)=\mathbf{r}_2(s)$ and get the following system of equations:

$$t=1-s, \tag{1}$$
$$3-t=s+2, \tag{2}$$
$$9+t^2=s^2. \tag{3}$$

Substituting (1) into (3) yields $9+(1-s)^2=s^2$ which is equivalent to $10=2s$. Hence $s=5$ which implies that $t=-4$ and so the intersection point is

$\mathbf{r}_1(-4) = (-4, 7, 25) = \mathbf{r}_2(5)$. To find the angle of intersection we need the angle between the unit tangent vectors $\mathbf{T}_1(t)$ and $\mathbf{T}_2(s)$ for the curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ at the point of intersection. Since $\mathbf{r}_1'(t) = \langle 1, -1, 2t\rangle$ and $\mathbf{r}_2'(s) = \langle -1, 1, 2s\rangle$, the acute angle between $\mathbf{T}_1(-4)$ and $\mathbf{T}_2(5)$ is

$$\cos\theta = |\mathbf{T}_1(-4)\cdot\mathbf{T}_2(5)| = \frac{|\mathbf{r}_1'(-4)\cdot\mathbf{r}_2'(5)|}{|\mathbf{r}_1'(-4)||\mathbf{r}_2'(5)|} = \frac{|-1-1-80|}{\sqrt{66}\sqrt{102}} = \frac{82}{\sqrt{66}\sqrt{102}}.$$

3. Given a smooth curve $\mathbf{r}(t)$, write $\mathbf{T}(t)$ for the unit tangent vector, $\mathbf{N}(t)$ for the principal unit normal vector, and $\mathbf{B}(t)$ for the unit binormal vector, each of which correspond to $\mathbf{r}(t)$. What can be said about their magnitudes? What can be said about the dot product between any two of them? What can be said about the cross product between any two of them?

**Solution:** Since these vectors are all *unit* vectors we know they have magnitude equal to one. This also implies that their dot product with themselves is equal to one. As they are orthonormal to each other, their dot product with one another is zero. For the cross product, we have that $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$ by definition. This implies $-\mathbf{B}(t) = \mathbf{N}(t)\times\mathbf{T}(t)$. Likewise, by the same reasoning used to justify the corresponding equations for the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, one has that

$$\begin{aligned} \mathbf{N}(t) &= \mathbf{B}(t)\times\mathbf{T}(t), \quad & -\mathbf{N}(t) &= \mathbf{T}(t)\times\mathbf{B}(t),\\ \mathbf{T}(t) &= \mathbf{N}(t)\times\mathbf{B}(t), \quad & -\mathbf{T}(t) &= \mathbf{B}(t)\times\mathbf{N}(t). \end{aligned}$$

4. Find both the tangential component $a_{\mathbf{T}}$ and normal component $a_{\mathbf{N}}$ of acceleration for the curve $\mathbf{r}(t) = \langle t^2, \cos t, \sin t\rangle$ at the initial point $(0, 1, 0)$.

**Solution:** First observe the curve intersects this point at $t = 0$ only. Hence we need to compute

$$a_{\mathbf{T}} = \frac{\mathbf{r}'(0)\cdot\mathbf{r}''(0)}{|\mathbf{r}'(0)|} \quad\text{and}\quad a_{\mathbf{N}} = \frac{|\mathbf{r}'(0)\times\mathbf{r}''(0)|}{|\mathbf{r}'(0)|}.$$

Since $\mathbf{r}'(t) = \langle 2t, -\sin t, \cos t\rangle$ and $\mathbf{r}''(t) = \langle 2, -\cos t, -\sin t\rangle$, we see that

$$\mathbf{r}'(0) = \langle 0, 0, 1\rangle \quad\text{and}\quad \mathbf{r}''(0) = \langle 2, -1, 0\rangle.$$

Thus $\mathbf{r}'(0)\cdot\mathbf{r}''(0) = 0$ and $|\mathbf{r}'(0)| = 1$ which shows that $a_{\mathbf{T}} = 0$ at $t = 0$. We can also compute that $\mathbf{r}'(0)\times\mathbf{r}''(0) = \langle 1, 2, 0\rangle$ and so $a_{\mathbf{N}} = \sqrt{5}$ at $t = 0$.