Warm-up — 18 November 2021

1. Evaluate
$$\int_C xe^{yz}\, ds,$$
where C is the line segment from $(0,0,0)$ to $(1,2,3)$.

Solution: The parameterization of this line is
$$\langle x(t), y(t), z(t)\rangle = \mathbf{r}(t) = (1-t)\langle 0,0,0\rangle + t\langle 1,2,3\rangle = \langle t, 2t, 3t\rangle.$$
Hence we compute
$$\int_C xe^{yz}\, ds = \int_0^1 te^{(2t)(3t)}\sqrt{1^2+2^2+3^2}\, dt = \boxed{\frac{\sqrt{14}}{12}\left(e^6-1\right).}$$

2. Evaluate
$$\int_C x^4\, dx + xy\, dy,$$
where C is the positively oriented triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Solution: By Green's theorem,
$$\int_C x^4\, dx + xy\, dy = \iint\limits_D \left(\frac{\partial}{\partial x}[xy] - \frac{\partial}{\partial y}[x^4]\right)\, dA = \iint\limits_D y\, dA$$
where D is the triangular region enclosed by C. Hence we get
$$\iint\limits_D y\, dA = \int_0^1\int_0^{1-x} y\, dy\, dx = \frac{1}{2}\int_0^1 (1-x)^2\, dx = \frac{1}{2}\left[\frac{u^3}{3}\right]_{u=0}^{u=1} = \boxed{\frac{1}{6}.}$$

3. Evaluate
$$\int_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4-1})\, dy,$$
where C is the positively oriented circle $x^2+y^2=9$.

Solution: By Green's theorem,

$$\int_C (3y - e^{\sin x})\ dx + (7x + \sqrt{y^4 - 1})\ dy = \iint_D (7-3)\ dA$$

where D is the disk bounded by C. Hence we get

$$\iint_D 4\ dA = \int_0^{2\pi} \int_0^3 4r\ dr\ d\theta = \boxed{36\pi.}$$

4. Find the work done by the force field $\mathbf{F}(x, y) = \langle x^2, -xy \rangle$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ with $0 \le t \le \pi/2$.

Solution: We have that

$$\begin{aligned} \text{Work} &= \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^{\frac{\pi}{2}} -2\cos^2 t \sin t\ dt \\ &= -\int_0^1 2u^2\ du \\ &= \left[-\frac{2}{3}u^3\right]_{u=0}^{u=1} = \boxed{-\frac{2}{3}.} \end{aligned}$$