Warm-up — 2 December 2021

1. Given a vector field $\mathbf{F} = \langle P, Q, R \rangle$ where P, Q, and R are functions of the variables x, y, and z with partial derivatives, give expressions for curl $\mathbf{F}$ and div $\mathbf{F}$.

Solution: The divergence of $\mathbf{F}$ is

$$\boxed{\text{div } \mathbf{F} = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R.}$$

The curl of $\mathbf{F}$ is

$$\boxed{\text{curl } \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle,}$$

where, for example, R_y denotes $\frac{\partial}{\partial y}R$.

2. How can you quickly tell that $\mathbf{F}(x, y, z) = \langle xz, xyz, -y^2 \rangle$ is not conservative? In addition, compute both div $\mathbf{F}$ and div curl $\mathbf{F}$.

Solution: Notice that curl $\mathbf{F} = \langle -2y - xy, x, yz \rangle$. If $\mathbf{F}$ were a conservative vector field then we would have that curl $\mathbf{F} = \langle 0, 0, 0 \rangle$, which is doesn't, therefore $\mathbf{F}$ cannot be conservative. Now use the expression for div $\mathbf{F}$ given above to compute that

$$\boxed{\text{div } \mathbf{F} = z + xz} \quad \text{and} \quad \boxed{\text{div curl } \mathbf{F} = -y + y = 0.}$$

You will revisit the latter computation in LQ34.

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

Solution: The curve C bounds an elliptical region S in the plane $y+z = 2$, so we can apply Stokes' theorem. First compute

$$\begin{aligned} \text{curl } \mathbf{F} &= \left\langle \frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(x), \frac{\partial}{\partial z}(-y^2) - \frac{\partial}{\partial x}(z^2), \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y^2) \right\rangle \\ &= \langle 0 - 0, 0 - 0, 1 - (-2y) \rangle \\ &= \langle 0, 0, 1 + 2y \rangle. \end{aligned}$$

By orienting S upward, its boundary curve C has positive orientation. The projection of S onto the xy-plane is the disk $x^2 + y^2 \leq 1$. Hence by Stokes' theorem we compute

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta) r \; dr \; d\theta \\ &= \frac{1}{6} \int_0^{2\pi} (3 + 4 \sin\theta) \; d\theta \\ &= \boxed{\pi.} \end{aligned}$$

4. Evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle xz, yz, xy \rangle$ and S is the part of the sphere $x^2+y^2+z^2 = 4$ that lies inside the cylinder $x^2+y^2 = 1$ and above the xy-plane.

Solution: Notice that the curve of intersection between the sphere and the cylinder bounds the solid S. We may solve for this curve C by setting the surface equations equal to each other to get that $x^2 + y^2 = 1$ and $z = \sqrt{3}$. Now parameterize the curve C by $\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$ for $0 \leq t \leq 2\pi$. Hence by Stokes' theorem we have that

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \; dt.$$

To compute the line integral in the last equation we notice that

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle \sqrt{3} \cos t, \sqrt{3} \sin t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle = 0.$$

Thus the corresponding line integral is zero and so overall we have shown

$$\boxed{\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0.}$$