Warm-up — 2 September 2021

1. Given arbitrary points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, write an equation that describes all the points (x, y, z) that are equidistant from both A and B.

Solution: The midpoint of the line segment joining A and B is given by

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

This point is certainly equidistant from both A and B. Moreover, so are all the points which lie on the plane P that is orthogonal to the vector

$$AB = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

and passes through the point M. The equation of the plane P is given by

$$\begin{aligned} 0 &= AB \cdot \left\langle x - \frac{x_1 + x_2}{2}, y - \frac{y_1 + y_2}{2}, z - \frac{z_1 + z_2}{2}\right\rangle \\ &= (x_2 - x_1)\left(x - \frac{x_1 + x_2}{2}\right) + \ldots + (z_2 - z_1)\left(z - \frac{z_1 + z_2}{2}\right) \\ &= (x_2 - x_1)x - \frac{x_2^2 - x_1^2}{2} + \ldots + (z_2 - z_1)z - \frac{z_2^2 - z_1^2}{2}. \end{aligned}$$

Moving the constant terms to the other side gives the equation

$$\boxed{(x_2 - x_1)x + (y_2 - y_1)y + (z_2 - z_1)z = \frac{1}{2}(x_2^2 + y_2^2 + z_2^2 - x_1^2 - y_1^2 - z_1^2)}$$

and this provides a quick method for solving problems like `Xronos HW01#9`.

For example, if $A = (1, -5, 1)$ and $B = (3, 3, -5)$ then our formula gives

$$(3 - 1)x + (3 - (-5))y + (-5 - 1)z = \frac{1}{2}(3^2 + 3^2 + (-5)^2 - 1^2 - (-5)^2 - 1^2).$$

Hence $2x + 8y - 6z = 8$ is an equation whose solutions (x, y, z) are precisely the points equidistant from these particular points A and B.

2. Compute the area of the triangle and the parallelogram spanned by two arbitrary vectors $a = \langle x_1, y_1, z_1\rangle$ and $b = \langle x_2, y_2, z_2\rangle$.

Solution: First note that the area of the triangle spanned by a and b is exactly half the area of the parallelogram that they span, the latter area being equal to $|a \times b|$. We can compute $a \times b$ by taking the determinant

$$\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = (y_1z_2 - y_2z_1)\mathbf{i} + (x_2z_1 - x_1z_2)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k},$$

which implies $a \times b = \langle y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1\rangle$. Hence the area of the parallelogram spanned by the vectors a and b is

$$\boxed{|a \times b| = \sqrt{(y_1z_2 - y_2z_1)^2 + (x_2z_1 - x_1z_2)^2 + (x_1y_2 - x_2y_1)^2},}$$

and the area of the triangle they span is $\frac{1}{2}|a \times b|$. This gives a quick method for solving exercises like `LQ4#2`.

For example, if $a = \langle -1, 1, 2\rangle$ and $b = \langle 2, 0, 1\rangle$ then our formula gives

$$|a \times b| = \sqrt{(1 \cdot 1 - 0 \cdot 2)^2 + (2 \cdot 2 - (-1) \cdot 1)^2 + (-1 \cdot 0 - 2 \cdot 1)^2},$$

which implies the area of the parallelogram spanned by these two vectors is $\sqrt{1^2 + 5^2 + (-2)^2} = \sqrt{30}$ and the area of the triangle they span is $\sqrt{30}/2$.

3. What can you say about the relationship between the plane P given by the equation $2x-2y+8z = 10$ and the line L defined parametrically by $\langle 1+t, 2-t, 4t\rangle$?

Solution: By inspection, the line L is parallel to the vector $\langle 1, -1, 4\rangle$. Likewise, the normal vector of the plane P is $\langle 2, -2, 8\rangle = 2 \cdot \langle 1, -1, 4\rangle$. Hence the normal vector of the plane P is parallel to the line L and therefore P is orthogonal to L. (This exercise is similar to `LQ5#3`.)

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4. Given any two vectors a and b, write a formula for the scalar and vector projections of a onto b. (Try to draw a picture!) In particular, what do your formulas say whenever $a = \langle -1, 5, 1\rangle$ and $b = \langle 1, 2, -2\rangle$?

> **Solution:** The scalar projection of a onto b is the scalar given by
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> $$\frac{a \cdot b}{|b|}.$$
>
> The vector projection of a onto b is the vector given by
>
> $$\left(\frac{a \cdot b}{|b|^2}\right) b.$$
>
> So then if $a = \langle -1, 5, 1\rangle$ and $b = \langle 1, 2, -2\rangle$, the scalar projection of a onto b is $7/3$ and the corresponding vector projection is $\langle 7/9, 14/9, -14/9\rangle$.

5. Let Q and R be distinct points and let L be the line which passes through both Q and R. Show that for any point P not on the line L, the shortest distance from P to L is exactly
$$\frac{|a \times b|}{|a|},$$
where a is the vector from Q to R and b is the vector from Q to P. (Try to draw a picture!) Use this result to find the shortest distance from the point $(3, 3, -1)$ to the line defined parametrically by $\langle 1 - t, 1 + t, -2 - 2t\rangle$.

> **Solution:** Suppose that Q, R, L, and P are given. Let a be the vector from Q to R and let b be the vector from Q to P. If we write d for the shortest distance between P and L and θ for the angle between the vectors a and b, it follows that
> $$d = |b| \sin\theta = \frac{|a \times b|}{|a|}.$$
> In particular, let L be given by $\langle 1 - t, 1 + t, -2 - 2t\rangle$ and consider the point $P = (3, 3, -1)$. One way to see that P is not on L is the nonexistence of a t such that $3 = 1 \pm t$. Now get distinct points Q and R on L by choosing two different values of t, say $Q = (1, 1, -2)$ and $R = (0, 2, -4)$. In this case, we have $a = \langle -1, 1, -2\rangle$ and $b = \langle 2, 2, 1\rangle$. Use the determinant method to compute $a \times b = \langle 5, -3, -4\rangle$. Hence the shortest distance from P to L is
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> $$\frac{|a \times b|}{|a|} = \frac{\sqrt{5^2 + (-3)^2 + (-4)^2}}{\sqrt{(-1)^2 + 1^2 + (-2)^2}} = \frac{\sqrt{50}}{\sqrt{6}} = \boxed{\frac{5\sqrt{3}}{3}}.$$