Warm-up — 21 October 2021

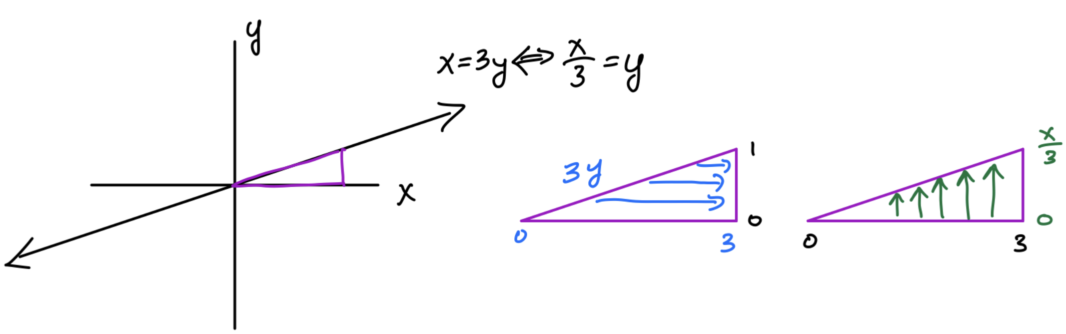
1. Show how you would go about computing

$$\int_0^1 \int_{3y}^3 \exp(x^2)\, dx\, dy.$$

Solution: We need to reverse the order of integration. To do so, we look at the bounds of the integral to determine the domain of integration. (Sketch the picture!) In this case we notice that as y varies between 0 and 1, the x component varies between $3y$ and 3. Hence we are integrating over the region bounded by the line $x = 3y$, the line $x = 3$ and the x-axis. Notice that as x varies between 0 and 3, the y component varies between 0 and $x/3$. Since the function $\exp(x^2)$ is continuous in the region we can say that

$$\begin{aligned}
\int_0^1 \int_{3y}^3 \exp(x^2)\, dx\, dy &= \int_0^3 \int_0^{\frac{x}{3}} \exp(x^2)\, dy\, dx \\
&= \int_0^3 \left[y \exp(x^2)\right]_{y=0}^{y=\frac{x}{3}}\, dx \\
&= \frac{1}{3} \int_0^3 x \exp(x^2)\, dx \\
&= \frac{1}{6} \int_0^9 \exp(u)\, du \quad (\text{where } u = x^2) \\
&= \frac{\exp(9) - 1}{6}
\end{aligned}$$

(See the sketch below.)



2. Find the volume of the solid S that is bounded by $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

Solution: Since S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0, 2] \times [0, 2]$ in the xy-plane, it follows that

$$\begin{aligned}
\text{the volume of } S &= \iint_R z \, dA \\
&= \int_0^2 \int_0^2 (16 - x^2 - 2y^2) \, dx \, dy \\
&= \int_0^2 \left[16x - \frac{x^3}{3} - 2y^2 x \right]_{x=0}^{x=2} dy \\
&= \int_0^2 \left(\frac{88}{3} - 4y^2 \right) dy \\
&= \left[\frac{88}{3} y - \frac{4}{3} y^3 \right]_{y=0}^{y=2} \\
&= \frac{176 - 32}{3} \\
&= 48.
\end{aligned}$$

3. Let D be the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. Sketch D in the xy-plane, then setup an integral to evaluate

$$\iint_D xy \, dA.$$

Solution: It's crucial to sketch the domain for these problems. By looking at our sketch we notice that we would definitely prefer to integrate with respect to x and then y, because as x varies between -3 and 5 we have a different behavior for the y component based on whether x is less than or greater than -1. So we would need to split the integral up into two separate ones and this seems like more work. Instead, observe that whenever y varies between -2 and 4 we simply have that the x component varies between the parabola $y^2/2 - 3$ and the line $y + 1$. Hence

$$\iint_D xy \, dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy \, dx \, dy.$$

and you're now in a good position to evaluate the integral.