Warm-up — 23 September 2021

1. Describe the level curves for $f(x, y) = \log(3x + 9y)$ and the level surfaces for $g(x, y, z) = y^2 + z^2$.

> **Solution:** If $k = \log(3x + 9y)$ then $3x + 9y = e^k$, which is just the equation of a line. So the level curves for $f(x, y)$ is just a family of parallel lines.
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> If $k = y^2 + z^2$ then we recognize this as a circular cylinder about the x-axis, so the level surfaces for $g(x, y, z)$ is a family of cylinders about the x-axis with varying radius.

2. Determine the domain and range of the function

$$f(x, y) = \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 4} - 2}.$$

Is $f(x, y)$ continuous on its entire domain? Pretend that you really wanted to extend this function to one that is continuous at every point in the plane. How would you accomplish this task?

> **Solution:** Notice that $\sqrt{x^2 + y^2 + 4} - 2 = 0$ if and only if $(x, y) = (0, 0)$. Now for $(x, y) \neq (0, 0)$ we can write
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> $$f(x, y) = \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 4} - 2}\left(\frac{\sqrt{x^2 + y^2 + 4} + 2}{\sqrt{x^2 + y^2 + 4} + 2}\right) = \sqrt{x^2 + y^2 + 4} + 2. \quad (\dagger)$$
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> We can see from the expression of $f(x, y)$ in ($\dagger$) that $f(x, y)$ is both defined and continuous whenever $(x, y) \neq (0, 0)$, and that $f(x, y)$ has range $(4, \infty)$. If you want to extend $f(x, y)$ by assigning a value to the origin in such a way that the resulting function is continuous at $(0, 0)$ then we first need to know the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ exists. But since $(x, y) \neq (0, 0)$ as (x, y) approaches $(0, 0)$ we may compute the limit as
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> $$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2 + 4} + 2 = \sqrt{4} + 2 = 4.$$
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> So the limit exists and if we wish to extend $f(x, y)$ to a continuous function we may simply define $g(x, y) = f(x, y)$ for $(x, y) \neq (0, 0)$ and set $g(0, 0) = 4$.

3. Show that

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

does not exist.

Solution: Write $f(x, y) = (xy^2)/(x^2 + y^4)$. As (x, y) approaches the origin along the y-axis we have $x = 0$ and so $f(x, y) = f(0, y) = 0$ along this line, meaning that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the y-axis. But if we consider (x, y) approaching the origin along the curve $x = y^2$ we get that $f(x, y) = f(y^2, y) = y^4/(2y^4) = 1/2$ along this curve. Hence $f(x, y) \to 1/2$ as $(x, y) \to (0, 0)$ along the parabola $x = y^2$. Since we have found different paths giving different limiting values we may conclude the limit does not exist.

4. Compute both of the following limits:

$$\lim_{(x,y)\to(0,0)} \frac{6xy}{\sqrt{x^2+y^2}} \quad \text{and} \quad \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}.$$

Solution: Convert to polar coordinates so that $x = r\cos\theta$ and $y = r\sin\theta$. Then $x^2 + y^2 = r^2$ and since we may take $r > 0$ for a suitable choice of θ, this implies that $\sqrt{x^2+y^2} = \sqrt{r^2} = r$. Now $6xy = 6r^2\cos\theta\sin\theta = 3r^2\sin 2\theta$. As $(x, y) \to (0, 0)$ we have that $r \to 0$, hence

$$\lim_{(x,y)\to(0,0)} \frac{6xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{3r^2\sin 2\theta}{r} = \lim_{r\to 0} 3r\sin 2\theta.$$

Now since $-1 \le \sin 2\theta \le 1$ for any θ, the fact that $3r > 0$ implies that $-3r \le 3r\sin 2\theta \le 3r$. Since $\pm 3r \to 0$ as $r \to 0$, we may conclude by squeeze theorem that $3r\sin 2\theta \to 0$ as $r \to 0$ and this shows

$$\lim_{(x,y)\to(0,0)} \frac{6xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} 3r\sin 2\theta = 0.$$

For the second limit we first note that

$$\frac{x^3+y^3}{x^2+y^2} = \frac{x^3}{x^2+y^2} + \frac{y^3}{x^2+y^2}.$$

We can show that both terms on the right-side above tend to 0 as (x, y) approaches the origin by a similar method as before. In polar coordinates with $r > 0$ we have

$$\frac{x^3}{x^2+y^2} = \frac{r^3\cos^3\theta}{r^2} = 3r\cos^3\theta.$$

Now since $-1 \le \cos\theta \le 1$ it follows that $-1 \le \cos^3\theta \le 1$ and therefore $-3r \le 3r\cos^3\theta \le 3r$ since $3r > 0$. But since $\pm 3r \to 0$ as $r \to 0$ we may conclude by squeeze theorem that $3r\cos^3\theta \to 0$ as $r \to 0$ and this shows

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} 3r\cos^3\theta = 0.$$

By similar reasoning, $y^3/(x^2+y^2) \to 0$ as $(x,y) \to 0$ and so both these limits exist. Hence the limit of their sum exists and is equal to the sum of their limits, i.e.,

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} + \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0+0 = 0.$$