Warm-up — 26 August 2021

1. For any real number x, the square root of x^2 equals ____________.

> **Solution:** The answer is $|x|$. If $x \geq 0$ then $\sqrt{x^2} = x$. On the other hand, if $x < 0$ then $\sqrt{x^2} = -x$. This is precisely the definition of $|x|$.

2. Given real-valued differentiable functions f, g, and h in a single variable x, write a formula for the derivative $(fgh)' = \frac{d}{dx}(fgh)$.

> **Solution:** Applying the ordinary Product Rule for derivatives to the functions f and gh gives
>
> $$(fgh)' = [(f)(gh)]' = f \cdot (gh)' + f' \cdot gh.$$
>
> Now applying the same result to the functions g and h gives
>
> $$(fgh)' = f(gh' + g'h) + f'gh = \boxed{fgh' + fg'h + f'gh.}$$

3. Compute $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$.

> **Solution:** We will use the method of integration by parts. By setting $u = x$ and $dv = (x-1)^{-1/2}dx$, the integral becomes $\int u\,dv = uv - \int v\,du$ after substitution. Hence
>
> $$\begin{aligned} \int \frac{x}{\sqrt{x-1}}\,dx &= 2x\sqrt{x-1} - 2\int (x-1)^{1/2}\,dx \\ &= 2x\sqrt{x-1} - \frac{4}{3}(x-1)^{3/2} + C \\ &= 2\sqrt{x-1}\left(x - \frac{2}{3}(x-1)\right) + C \\ &= \boxed{\frac{2}{3}\sqrt{x-1}(x+2) + C.} \end{aligned}$$

Remark: Rather than appealing to integration by parts, this exercise can instead be solved by using the substitution $u = x - 1$, since then

$$\begin{aligned}\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du &= \int (u^{1/2} + u^{-1/2})\,du \\ &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\ &= \boxed{\frac{2}{3}\sqrt{u}(u+3) + C}\end{aligned}$$

and the result follows.

4. Write a formula for the shortest distance from the point (x, y, z) to the y-axis. What does this tell you about the distance between $(2, -3, 4)$ and the y-axis? Precisely what would you type into Canvas to record your solution?

Solution: The point on the y-axis closest to the point (x, y, z) is $(0, y, 0)$, and the distance between these points is

$$\sqrt{(x-0)^2 + (y-y)^2 + (z-0)^2} = \sqrt{x^2 + z^2}.$$

In particular, this says that the shortest distance between the point $(2, -3, 4)$ and the y-axis is

$$\sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}.$$

If this were an LQ assignment on Canvas you would want to enter: `2sqrt(5)`.

5. Given two vectors u and v, how do we check if they are parallel to each other? How do we check if they are perpendicular to each other? Give a vector v parallel to $u = \langle 2, 10, -4\rangle$ that points in the *opposite* direction.

Solution: The vectors u and v are parallel if and only if $u = c \cdot v$ for some real number c, i.e., whenever they are scalar multiples of one another. The vectors u and v are perpendicular if and only if $u \cdot v = 0$, i.e., their dot-product equals zero. To find such a vector v for $u = \langle 2, 10, -4\rangle$, pick any real number $c < 0$ and let $v = c \cdot u = \langle 2c, 10c, -4c\rangle$. For example, $v = \langle -1, -5, 2\rangle$ works, corresponding to $c = -1/2$.