**Warm-up** — 30 September 2021

1. Find both $\partial z/\partial x$ and $\partial z/\partial y$ for $z = f(x, y)$ defined implicitly by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1. \qquad (\dagger)$$

(*Hint*: Perhaps apply a theorem to save yourself some work.)

**Solution:** We could find $\partial z/\partial x$ by performing implicit differentiation with respect to $x$ on equation ($\dagger$) to get

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0,$$

which we may solve for $\partial z/\partial x$ to conclude

$$\boxed{\frac{\partial z}{\partial x} = -\frac{3x^2 + 6yz}{3z^3 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}.}$$

Likewise, we could do the same for $\partial z/\partial y$ and be done. Alternatively, one could solve this exercise by appealing to the **Implicit Function Theorem**. Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$ and notice that $F$ is defined at all points in space. Observe $F(0, 0, 1) = 0$ and

$$F_z(x, y, z) = 3z^2 + 6xy,$$

which implies $F_z(0, 0, 1) = 3 \neq 0$ and shows $F_z$ is continuous at every point in space. As $F_x(x, y, z) = 3x^2 + 6yz$ and $F_y(x, y, z) = 3y^2 + 6xz$ are also continuous at every point in space, the theorem applies and one has

$$\boxed{\frac{\partial z}{\partial y} \overset{!}{=} -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}.}$$

2. Show that

$$f(x, y) = xe^{xy}$$

is differentiable at $(1, 0)$. What can you say about $f(1.1, -0.1)$?

> **Solution:** As $f_x(x, y) = e^{xy}(1 + xy)$ and $f_y(x, y) = x^2e^{xy}$ are continuous at every point in the plane we know that $f$ is differentiable everywhere. Hence
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> $$f(1.1, -0.1) \approx f(1, 0) + f_x(1, 0)(1.1-1) + f_y(1, 0)(-0.1-0) = 1 + 0.1 - 0.1 = 1.$$
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> More generally we have the linear approximation $xe^{xy} \approx x + y$ near $(1, 0)$.

3. Suppose you have a right circular cone that stands at a height of 25 meters with a radius of 10 meters at its base, where both measurements are accurate to within 0.2 meters. Use differentials to estimate the maximum error in the calculated volume of the cone. (*Hint*: $V = \pi r^2h/3$.)

> **Solution:** The measurements being accurate to within 0.2 meters means the height is $25 \pm 0.1$ meters and the base radius is $10 \pm 0.1$ meters. The volume of such a cone is $V = \pi r^2h/3$ and therefore
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> $$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh,$$
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> which shows that the error in our calculated volume is no more than
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> $$\frac{2\pi \cdot 10 \cdot 25}{3}(0.1) + \frac{\pi(10)^2}{3}(0.1) = \frac{500\pi}{30} + \frac{100\pi}{30} = \frac{600\pi}{30} = 20\pi \text{ cubic meters.}$$

4. Given $w = x^4y + y^2z^3$ with $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial w/\partial s$ whenever $r = 2$, $s = 1$, and $t = 0$.

> **Solution:** By the chain rule we know that
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> $$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds} + \frac{\partial w}{\partial z}\frac{dz}{ds} \\ &= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t).\end{aligned}$$
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> So when $r = 2$, $s = 1$, and $t = 0$ we get $(64)(2) + (16)(4) + (0)(0) = 192$.