**Warm-up** — 4 November 2021

1. Find the volume of the solid bounded by $z = 3+\sqrt{9-x^2-y^2}$ and $z = \sqrt{x^2+y^2}$.

**Solution:** Write $E$ for the solid in question. Notice

$$z = 3+\sqrt{9-x^2-y^2} \quad \text{if and only if} \quad x^2+y^2+(z-3)^2 = 9,$$

which is a sphere of radius three centered at the point $(0, 0, 3)$. The cone $z = \sqrt{x^2+y^2}$ intersects this sphere along its equator in a circle of radius three on the plane $z = 3$, so $E$ resembles an ice cream cone containing a single scoop of ice cream. Of course, we could note that the upper half of this sphere has volume

$$\frac{1}{2}\left(\frac{4\pi 3^3}{3}\right) = \frac{36\pi}{2} = 18\pi$$

and this right-cone with height three and base radius three has volume

$$\frac{\pi 3^2 \cdot 3}{3} = 9\pi,$$

leading to the conclusion that $E$ has volume $18\pi+9\pi = 27\pi$. But this wouldn't help us if we were asked to setup an integral to find the volume, so we'll just use this to check our answer. As the solid encircles the $z$-axis we have that $0 \le \theta \le 2\pi$. Using the substitution $x = \rho\sin\phi\cos\theta$ and $y = \rho\sin\phi\sin\theta$ we have that $x^2+y^2 = \rho^2\sin^2\phi$. Moreover, $z = \rho\cos\phi$ and thus the cone satisfies

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi} = \rho\sin\phi.$$

This implies $\cos\phi = \sin\phi$ whenever $\rho \neq 0$ and hence $0 \le \phi \le \pi/4$. Furthermore, our first equation above implies $\rho^2 - 6\rho\cos\phi = 0$, showing that $0 \le \rho \le 6\cos\phi$. So then we have

$$\begin{aligned}
\text{the volume of } E &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{6\cos\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta \\
&= 72\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\phi\cos^3\phi \, d\phi \, d\theta \\
&= 72\int_0^{2\pi} d\theta \cdot \int_0^{\frac{\sqrt{2}}{2}} (u-u^3)du \\
&= 144\pi\left[\frac{u^2}{2} - \frac{u^4}{4}\right]_{u=0}^{u=\frac{\sqrt{2}}{2}} = 144\pi\cdot\frac{3}{16} = \boxed{27\pi.}
\end{aligned}$$

2. For the region $R = \{(x, y) : -\sqrt{4-y^2} \le x \le -y \text{ and } 0 \le y \le \sqrt{2}\}$, compute

$$\iint_R 16\cos(x^2+y^2)\, dA.$$

**Solution:** Sketch the region $R$ in the $xy$-plane and observe that the line $y = \sqrt{2}$ intersects the half circle $x = -\sqrt{4-y^2}$ at the point $(-\sqrt{2}, \sqrt{2})$. The line $x = -y$ passes through this point and the origin with angle $3\pi/4$, and since $-\sqrt{4-y^2} \le x \le -y$ for $0 \le y \le \sqrt{2}$, it follows that $R$ is the "pie slice" bounded by the lines $y = 0$ and $x = -y$ together with the half circle $x = -\sqrt{4-y^2}$. Therefore

$$R = \{(r, \theta) : 0 \le r \le 2 \text{ and } 3\pi/4 \le \theta \le \pi\}$$

and so

$$\begin{aligned}
\iint_R 16\cos(x^2+y^2) dA &= \int_{\frac{3\pi}{4}}^{\pi} \int_0^2 16r\cos(r^2)\, dr\, d\theta \\
&= 8\int_{\frac{3\pi}{4}}^{\pi} d\theta \cdot \int_0^4 \cos u\, du \\
&= 8\left[\pi - \frac{3\pi}{4}\right] \cdot [\sin(4) - \sin(0)] \\
&= \boxed{2\pi\sin(4).}
\end{aligned}$$

3. Evaluate

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

**Solution:** Inspect the bounds of the above integral to determine that the solid of integration is the portion of a sphere of sphere with radius four centered at the origin that lies in the first quadrant. This solid can be described in terms of spherical coordinates as

$$\{(\rho, \theta, \phi) : 0 \le \rho \le 4 \text{ and } 0 \le \theta \le \pi/2 \text{ and } 0 \le \phi \le \pi/2\}.$$

Upon switching to spherical coordinates, the integral becomes

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^4 \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{\pi}{2}\left[\cos(0) - \cos\left(\frac{\pi}{2}\right)\right]\left[\frac{4^4}{4} - \frac{0^4}{4}\right] = \boxed{32\pi.}$$