Warm-up — 7 October 2021

1. Identify the critical points of

$$f(x, y) = xy - x^2y - xy^2.$$

Identify and classify the critical points of

$$g(x, y) = (y^2 - 1)(e^x - 1).$$

Solution: The critical points of $f(x, y)$ are the points (a, b) where both $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of those does not exist. Observe that

$$f_x(x, y) = y - 2xy - y^2 \quad \text{and} \quad f_y(x, y) = x - x^2 - 2xy.$$

Hence $0 = f_x(x, y) = y(1 - 2x - y)$ if and only if $y = 0$ or $1 - 2x - y = 0$. Likewise, $0 = f_y(x, y) = x(1 - x - 2y)$ if and only if $x = 0$ or $1 - x - 2y = 0$. Considering all cases we find that both $f_x(x, y) = 0$ and $f_y(x, y) = 0$ at the points $(0, 0), (1, 0), (0, 1)$, and $(1/3, 1/3)$. As $f_x(x, y)$ and $f_y(x, y)$ are defined for all points in the plane it follows that these four points are the only critical points of $f(x, y)$.

We carry out the same procedure for $g(x, y)$ except this time we will go further and analyze the critical points. Notice

$$g_x(x, y) = (y^2 - 1)e^x \quad \text{and} \quad g_y(x, y) = 2y(e^x - 1),$$

which lead to

$$\begin{aligned} g_{xx}(x, y) &= (y^2 - 1)e^x, & g_{xy}(x, y) &= 2ye^x, \\ g_{yx}(x, y) &= 2ye^x, & g_{yy}(x, y) &= 2(e^x - 1). \end{aligned}$$

As $0 = g_x(x, y) = (y^2-1)e^x$ if and only if $y^2 = \pm 1$ and $0 = g_y(x, y) = 2y(e^x-1)$ if and only if $x = 0$ or $y = 0$, it follows that $(0, -1)$ and $(0, 1)$ are critical points of $g(x, y)$. Since $g_x(x, y)$ and $g_y(x, y)$ are defined on the whole plane, these two points are the only two critical points of $g(x, y)$. Now define

$$D(x, y) = \det \begin{bmatrix} g_{xx}(x, y) & g_{xy}(x, y) \\ g_{yx}(x, y) & g_{yy}(x, y) \end{bmatrix}$$

and compute

$$D(0, -1) = \det \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = -4 \quad \text{and} \quad D(0, 1) = \det \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = -4.$$

From this we can conclude that both $(0, -1)$ and $(0, 1)$ are saddle points.

2. Find a vector equation for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

Solution: Define $F(x, y, z) = z - x^2 - y^2$ and $G(x, y, z) = 4x^2 + y^2 + z^2 - 9$. The tangent line to the curve of intersection at $(-1, 1, 2)$ will be parallel to

$$\nabla F(-1, 1, 2) \times \nabla G(-1, 1, 2).$$

Compute that $\nabla F(-1, 1, 2) = \langle 2, -2, 1\rangle$ and $\nabla G(-1, 1, 2) = \langle -8, 2, 4\rangle$. Hence

$$\nabla F(-1, 1, 2) \times G(-1, 1, 2) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{bmatrix} = \langle -10, -16, -12\rangle,$$

and thus the parametric equation $\langle -1 - 10t, 1 - 16t, 2 - 12t\rangle$ describes the tangent line to the curve of intersection of these two surfaces at $(-1, 1, 2)$. (Note that any line parallel to this one which shares the point $(-1, 1, 2)$ is also a valid solution, e.g., $\langle -1 - 5t, 1 - 8t, 2 - 6t\rangle$.)

3. At which point(s) on the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the tangent plane parallel to the plane $x + 2y + z = 1$?

Solution: Define $F(x, y, z) = x^2+y^2+2z^2-1$. Then $\nabla F(a, b, c) = \langle 2a, 2b, 4c\rangle$ is a normal vector to the tangent plane at each point (a, b, c) on the surface. If we want this plane to be parallel to the plane $x + 2y + z = 1$ then we need their normal vectors to be parallel. Hence $\langle 2a, 2b, 4c\rangle = k\langle 1, 2, 1\rangle$ for some nonzero real number k. This implies $a = k/2$, $b = k$, and $c = k/4$. But for (a, b, c) to lie on the ellipsoid we must have that

$$1 = a^2 + b^2 + 2c^2 = (k/2)^2 + k^2 + 2(k/4)^2 = \frac{11}{8}k^2,$$

which occurs precisely when $k = \pm 2\sqrt{2/11}$. Therefore, the tangent planes to the given ellipsoid at the points

$$\left(-\sqrt{\frac{2}{11}}, -2\sqrt{\frac{2}{11}}, -\frac{1}{2}\sqrt{\frac{2}{11}}\right) \quad \text{and} \quad \left(\sqrt{\frac{2}{11}}, 2\sqrt{\frac{2}{11}}, \frac{1}{2}\sqrt{\frac{2}{11}}\right)$$

are parallel to the given plane.

4. If $f(x, y, z) = x^3 + yz$, show there is no unit vector $\mathbf{v}$ for which $D_{\mathbf{v}} f(1, 1, -1) = 4$.

Solution: Since $f(x, y, z)$ is differentiable at $(1, 1, -1)$ we know that

$$|D_{\mathbf{v}} f(1, 1, -1)| \leq |\nabla f(1, 1, -1)| = |\langle 3, -1, 1 \rangle| = \sqrt{11} < \sqrt{16} = 4$$

for any unit vector $\mathbf{v}$, hence no such vector exists.

5. Find three positive numbers whose sum is 100 and whose product is a maximum.

Solution: We wish to maximize xyz subject to the constraint that $x+y+z = 100$ and that x, y, and z are positive real numbers. Notice $z = 100 - x - y$, so define

$$f(x, y) = xy(100 - x - y) = 100xy - x^2y - xy^2$$

so that we are looking for the maximum value of $f(x, y)$. To do this we start by looking for the critical points of $f(x, y)$, which are the points where $f_x(x, y) = 0$ and $f_y(x, y) = 0$, or a point where one of these functions is not defined. We compute that

$$f_x(x, y) = 100y - 2xy - y^2 \quad \text{and} \quad f_y(x, y) = 100x - x^2 - 2xy,$$

so $0 = f_x(x, y) = y(100 - 2x - y)$ if and only if $y = 0$ or $100 - 2x - y = 0$. Likewise, $0 = f_y(x, y) = x(100 - x - 2y)$ if and only if $x = 0$ or $100 - x - 2y = 0$. Since we insist that x and y be positive, it follows that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ if and only if

$$100 - 2x - y = 0 \quad \text{and} \quad 100 - x - 2y = 0.$$

The first equation says that $y = 100 - 2x$ and by substituting this into the second equation we obtain that $x = 100/3$. It then follows that

$$y = 100 - 2(100/3) = 100/3 \quad \text{and} \quad z = 100 - 100/3 - 100/3 = 100/3.$$

Hence $x = y = z = 100/3$ is our solution and the maximum value of their product is $(100/3)^3 \approx 37,037$.