Warm-up — 9 September 2021

1. Write down the standard equation for each of the surfaces listed below:

Ellipsoid: ______ Hyperboloid (one sheet): ______ Hyperboloid (two sheets): ______
Cone: ______ Elliptic Paraboloid: ______ Hyperbolic Paraboloid: ______

Solution:

- Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
- Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$
- Hyperboloid (one sheet): $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$
- Hyperboloid (two sheets): $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
- Elliptic Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0.$
- Hyperbolic Paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0.$

2. Identify each of the following surfaces and be as descriptive as you wish:

a.) $x^2 + y^2 = 1$ b.) $y^2 + z^2 = 1$ c.) $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$
d.) $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$ e.) $4x^2 - y^2 + 2z^2 + 4 = 0$ f.) $x^2 + 2z^2 - 6x - y + 10 = 0.$

Solution:

a.) Cylinder about the z-axis with radius 1.

b.) Cylinder about the x-axis with radius 1.

c.) Ellipsoid ceneted about the origin with the constraints $-1 \leq x \leq 1$, $-2 \leq y \leq 2$, and $-3 \leq z \leq 3$.

d.) Hyperboloid about the z-axis with one sheet.

e.) Hyperboloid about the y-axis with two sheets.

f.) Elliptic paraboloid with tip at the point $(3, 1, 0)$.

3. For any real-valued nonzero vector v, what is the value of $|v \cdot v|$? What is the value of $|v \cdot v|$ whenever v is the zero vector?

Solution: Assuming v is real-valued and nonzero, we have

$$\frac{|v \cdot v|}{|v| \cdot |v|} = \cos 0 = 1$$

since the angle between v and itself is zero. Hence $\boxed{|v \cdot v| = |v|^2}$. If v is the zero vector then $v \cdot v = 0$ and so $|v \cdot v| = 0$.

4. Find the domain of the vector function

$$\mathbf{r}(t) = \left\langle \sqrt{2-t}, \frac{e^t - 5}{t}, \ln(t+1) \right\rangle .$$

Solution: The domain of $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ will be the intersection of the domains of the functions $x(t)$, $y(t)$, and $z(t)$. The domain of $\sqrt{2-t}$ is the set of real numbers t such that $2 - t \geq 0$. Equivalently, the real numbers t such that $t \leq 2$. Similarly, the domain of $(e^t - 5)t^{-1}$ is all *nonzero* real numbers. Lastly, the domain of $\ln(t + 1)$ is all real t such that $t + 1 > 0$, or equivalently, the real t such that $t > -1$. Hence the domain of $\mathbf{r}(t)$ in this case is $(-1, 0) \cup (0, 2]$.

5. Find the unit tangent vector at the point $(1, 0, 0)$ for the curve

$$\mathbf{r}(t) = \left\langle 1 + t^3, te^{-t}, \sin(2t) \right\rangle.$$

Solution: First notice that the point $(1, 0, 0)$ intersects the curve $\mathbf{r}(t)$ at $t = 0$. Now compute the derivative of $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ with respect to t by treating the functions $x(t)$, $y(t)$, and $z(t)$ independently to obtain

$$\mathbf{r}'(t) = \langle 3t^2, (1 - t)e^{-t}, 2\cos(2t)\rangle.$$

So to find the unit tangent vector at the point $(1, 0, 0)$ we compute

$$\frac{1}{|\mathbf{r}'(0)|}\mathbf{r}'(0) = \frac{1}{\sqrt{5}}\langle 0, 1, 2\rangle = \left\langle 0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right\rangle.$$

(Useful) 6. Show that $2\cos^2(t) = 1 + \cos(2t)$ for any real number t.

Solution: Recall that $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ for all real numbers α and β. Hence

$$\cos(2t) = \cos(t + t) = \cos(t)\cos(t) - \sin(t)\sin(t) = \cos^2(t) - \sin^2(t)$$

for any real number t. Now add $1 = \cos^2(t) + \sin^2(t)$ to this equation to obtain

$$1 + \cos(2t) = 2\cos^2(t),$$

and this is what we wished to show.

APPLICATION: To see how this identity is useful, consider

$$\int \cos^2(2t)dt.$$

This integral seems challenging at first glance, but the identity above yields

$$\begin{aligned}\int \cos^2(2t)dt &= \frac{1}{2}\int [1 + \cos(4t)]dt \\ &= \frac{1}{2}\left(t + \frac{\sin(4t)}{4}\right) + C.\end{aligned}$$