Warm-up — 14 October 2021

1. Prove that the rectangle with maximum area and fixed perimeter p is a square.

Solution: Suppose R is a rectangle with side lengths x and y and has a fixed perimeter p. Then the area of R is given by $f(x,y) = xy$, so we wish to find the maximum value of the surface $f(x,y)$ subject to the constraint $2x + 2y = p$. Then for $g(x,y) = 2x + 2y$, observe that

$$\nabla f(x,y) = \lambda \nabla g(x,y) \iff \langle y, x\rangle = \langle 2\lambda, 2\lambda\rangle \iff x = 2\lambda = y.$$

Hence the side lengths of R must be equal and so R is a square as advertised.

2. Find the extreme values of $f(x,y) = e^{-xy}$ on the region $x^2 + 4y^2 \leq 1$.

Solution: First compute

$$f_x(x,y) = -ye^{-xy} \quad \text{and} \quad f_y(x,y) = -xe^{-xy}.$$

Hence $0 = f_x(x,y)$ if and only if $y = 0$ and $0 = f_y(x,y)$ if and only if $x = 0$. Therefore $(0,0)$ is the only critical point of $f(x,y)$ and note that $f(0,0) = 1$. We will now find the extreme values of $f(x,y)$ subject to the constraint $x^2 + 4y^2 = 1$. To do so, write $g(x,y) = x^2 + 4y^2$ and now consider

$$\nabla f(x,y) = \lambda \nabla g(x,y) \iff \langle -ye^{-xy}, -xe^{-xy}\rangle = \langle 2\lambda x, 8\lambda y\rangle.$$

This sets up the following system of equations:

$$\begin{cases} -ye^{-xy} = 2\lambda x & \textbf{(1)} \\ -xe^{-xy} = 8\lambda y & \textbf{(2)} \\ x^2 + 4y^2 = 1 & \textbf{(3)}. \end{cases}$$

If $y = 0$ then $x = \pm 1$ by **(3)** but we would also have $x = 0$ by **(2)** and so this is not possible. Hence $y \neq 0$ and so we may divide **(1)** by y to obtain

$$-e^{-xy} = 2\lambda x / y. \qquad (\dagger)$$

Substituting this into **(2)** gives $x(2\lambda x/y) = 8\lambda y$ which implies $x^2 = 4y^2$ since one can now check that $\lambda \neq 0$ by **(1)** and **(2)**. Plugging this expression into **(3)** gives $8y^2 = 1$ so that $y = \pm 1/(2\sqrt{2})$ and therefore $x = \pm 1/\sqrt{2}$. It now follows that

$$f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \quad \text{and} \quad f\left(\mp\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4}$$

are the maxima and minima on the region, respectively.

3. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ is an ellipse. Find the points on this ellipse that are nearest and farthest from the origin.

Solution: The distance from a point (x, y, z) on the ellipse of intersection of these surfaces is

$$d(x, y, z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}.$$

So we wish to find the points (x, y, z) where $d(x, y, z)$ attains its maximum and minimum values. Note that this is equivalent to finding the points where

$$D(x, y, z) = x^2 + y^2 + z^2$$

attains its maximum and minimum values since the square-root is an increasing function, and the algebra will be nicer. So we find the extreme values of $D(x, y, z)$ subject to the two constraints $x+y+2z = 2$ and $z = x^2+y^2$. Write $F(x, y, z) = x + y + 2z$ and $G(x, y, z) = z - x^2 - y^2$. Then we aim to find the points (x, y, z) and scalars λ and μ such that

$$\nabla D(x, y, z) = \lambda \nabla F(x, y, z) + \mu \nabla G(x, y, z).$$

Computing the gradients, this gives the vector equation

$$\langle 2x, 2y, 2z \rangle = \langle \lambda, \lambda, 2\lambda \rangle + \langle -2\mu x, -2\mu y, \mu \rangle$$

and sets up the following system of equations:

$$\begin{cases} 2x = \lambda - 2x\mu & \textbf{(1)} \\ 2y = \lambda - 2y\mu & \textbf{(2)} \\ 2z = 2\lambda + \mu & \textbf{(3)} \\ x + y + 2z = 2 & \textbf{(4)} \\ z = x^2 + y^2 & \textbf{(5)}. \end{cases}$$

Equating λ in **(1)** and **(2)** leads to $(2x - 2y)(1 + \mu) = 0$. Hence we must have that either $\mu = -1$ or $x = y$. If $\mu = -1$ then $\lambda = 0$ by **(1)** and therefore $z = -1/2$ by **(3)**. But this is not possible since $z = x^2 + y^2 \geq 0$ by **(5)**. Hence $\mu \neq -1$ so we must have that $x = y$. Thus equations **(4)** and **(5)** simplify to $2x + 2z = 2$ and $z = 2x^2$, respectively. Together these say $1 = x + z = x + 2x^2$, which is equivalent to finding the solutions to $0 = 2x^2 + x - 1 = (2x - 1)(x + 1)$. So the two points to check are

$$D\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4} \quad \text{and} \quad D(-1, -1, 2) = 6.$$

Hence the ellipse gets as close as $\sqrt{3}/2$ and as far as $\sqrt{6}$ from the origin.