Answer the following problems. No calculators, formula sheets, or other aids are permitted. Please show all of your work. Each question is worth 5 points.

1. Let R be the region bounded by $2x + 3y = 2$, $x - y = 1$, $2x + 3y = 4$, $x - y = 4$. Use a transformation to evaluate the integral:

$$\iint_R \frac{2}{3}(x - y)\, dA$$

We let $u = 2x + 3y$ and $v = x - y$. Then it is clear that $2 \leq u \leq 4$ and $1 \leq v \leq 4$. Now we find the Jacobian of this transformation:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

I prefer computing the inverse as so:

$$J^{-1} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}] = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

Now $det(J^{-1}) = -2 - 3 = -5$, so $|det(J)| = 1/5$ Therefore, the transformed integral is:

$$\int_2^4 \int_1^4 \frac{2}{3} v (\frac{1}{5}) dv du$$

Evaluating yields: $\int_2^4 [\frac{v^2}{15}]_1^4 du = \int_2^4 du = [u]_2^4 = 4 - 2 = 2.$

2. Set up, but do NOT evaluate a double integral to find the area inside the circle $x^2 + (y - 3)^2 = 9$ but OUTSIDE the circle $x^2 + y^2 = 9$ in the first quadrant.

We use the standard polar transformation of $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Therefore our new equations are:

$r^2\cos(\theta)^2 + (r\sin(\theta) - 3)^2 = 9$ and $r^2 = 9$. This yields $r = 6\sin(\theta)$ and $r = 3$.

The inside circle is $r = 3$ and the outside circle is $r = 6\sin(\theta)$, and therefore $3 \leq r \leq 6\sin(\theta)$.

We note the determinant of the Jacobian in a standard polar transformation is simply r.

Now we find where the circles intersect: $3 = 6\sin(\theta)$ is true when $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. Since the area is restricted to the first quadrant, this means $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$.

Thus our integral looks like $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_3^{6\sin(\theta)} r dr d\theta$