Exam 2: Tuesday, Oct 15, @ ~~6:15 PM~~ 5:10 PM in FLINT 0050

Office Hours on ZOOM this week

Key concepts to know (by lecture):

Things to bring:
- ID
- pencil for scantron
- any remaining brain cells

Lecture 10: Domain & Range

Example - see quiz 3

Lecture 11: Limits & Continuity

- Be able to show a limit exists / does not exist
- Understand what makes a piecewise multivariable function continuous
- Example - see quiz 3 & Kronos 11 #10 & 12

Lecture 12: Partial Derivatives

- Example: see quiz 4

Lecture 13 & part of 16: Tangent Planes & Linear Approximations ★★★

- A lot of questions can be asked from this lecture
- Similar also to lecture 16
- See Spring 2018 #4, 8, 10, 16, FRQ #16-c,

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is the linear approx of $f(x,y)$ @ (a,b).

same as tangent plane formula

Lecture 14: Chain Rule

- Example: see quiz 4
- if we have a function z in terms of u and v, where u and v are in terms of x and y, then for example,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

Lecture 15: Directional Derivatives & Gradients * * *

- A lot of questions can be asked
  - Directional derivative of f @ (a,b) in the direction of a *unit vector* $\hat{u} = \langle u_1, u_2 \rangle$ is
    $D_{\hat{u}} f(a,b) = f_x(a,b)\,u_1 + f_y(a,b)\,u_2$
  - can also write unit vector as $\langle \cos\theta, \sin\theta \rangle$
  - Gradient $= \nabla f = \langle f_x, f_y \rangle$
  - Gradient tells us in what direction $\hat{u}$ $D_{\hat{u}} f(x,y)$ is smallest & largest (Spring 2018 #1)
  - $\underbrace{-|\nabla f(a,b)|}_{\text{"descent"}} \leq D_{\hat{u}} f(a,b) \leq \underbrace{|\nabla f(a,b)|}_{\text{"ascent"}}$

Lecture 16: Gradients & Tangent Planes (part 2)

- implicit functions: $f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$

hint: you can always use implicit *

· Lectures 17&18: Optimization (Lagrange Multipliers) ***

· $\nabla f = \lambda \nabla g$

· There will be an FRQ on this (per Dr. Shabazz)

· Ex: Spring 2018 FRQ #2

· Classifying Critical Points:

$D = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$D > 0$, $f_{xx}(a,b) < 0 \Rightarrow f(a,b)$ is local max

$D > 0$, $f_{xx}(a,b) > 0 \Rightarrow f(a,b)$ is local min

$D < 0 \Rightarrow f(a,b)$ is saddle point

$D = 0 \Rightarrow$ inconclusive

$x^2 + y^2 < 1$

In class, we will go over Spring 2018 FRQ #2, Review Sheet #11, 15, 26, X16 #8

Spring 2018 FRQ #2: Use the Method of Lagrange Multipliers to find the minimum value of

$$f(x,y,z) = 2x^2 + y^2 + 2z^2$$

on the plane $x - y + 3z = 6$.

Solution:

Review Sheet #11) Find an equation of the tangent plane to the surface $z = x\sin(xy)$ @ $(-1, 1, 0)$

Technically explicit, but I use

#15 let $f(x,y) = \ln(1 + xy)$

a) Find unit vectors that give direction of steepest ascent & steepest descent at (1, 2)

b) Find a unit vector that points in the direction of no change @ (1, 2)

Want vector perpendicular to gradient

Review #26) True or False:

26. True or False:

(1) There exists a function f with continuous second partial derivatives such that $f_x = x + y^2$ and $f_y = x - y^2$.

(2) If $f_x(a, b)$ and $f_y(a, b)$ both exist, then f is differentiable at (a, b).

(3) If $f(x, y)$ is differentiable, then the rate of change of f at the point (a, b) in the direction of $\vec{w}$ is $\nabla f(a, b) \cdot \cancel{\vec{w}}$. $\hat{w}$

(4) If $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then f must have a local maximum or minimum at (a, b).

(5) If $f(x, y)$ is differentiable and f has a local minimum at (a, b), then $D_{\vec{u}}f(a, b) = 0$ for any unit vector $\vec{u}$.

(1) False: $f_{xy} = 2y \quad f_{yx} = -2y$

(2) False: f must be continuous @ (a,b) 2/2

(3) False: w must be unit

(4) False: Saddles exist!

(5) True: $\nabla f = 0$

(3) $D_u f(a,b) = \nabla f(a,b) \cdot u$

$= 0 \cdot u = 0$

(4)

(5)

What does it mean to say f is differentiable?

f_x & f_y must exist AND f must be continuous

Consider NYTI L12

4. Let $f(x) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

Find $f_x(0,0)$ and $f_y(0,0)$. Is f continuous at $(0,0)$?

$f_x(0,0) = 0$, $f_y(0,0) = 0$, but f is not continuous: along $y = 0$ limit is 0, alo

$f_x = \begin{cases} \text{some quotient rule} & x,y = (0,0) \\ 0 & x,y = 0,0) \end{cases}$

$f_x(0,0) = 0$

$f_y(0,0) = 0$

but f is not continous!

X16 #8

Consider the surfaces $x^2 + z^2 = 20$ and $y^2 + z^2 = 25$.

Find symmetric equations of the tangent line to the curve of intersection @ $(2,3,4)$

LI T215 2022 Fall

10:30–4pm

Spring 2018 #9

$f(x,y) = 2x^2y + 2y^3 + 3y^2$

on disc $x^2+y^2 \leq 1$

$\underbrace{\text{boundary: } x^2+y^2=1}_{g(x,y)}$

f has two critical points $(0,0)$ and $(0,-1)$

w/ $f(0,0)=0$ and $f(0,-1)=1$

Boundary

$\nabla f = \lambda \nabla g$

$\nabla f = \langle 4xy, 2x^2+6y^2+6y \rangle$

$\nabla g = \langle 2x, 2y \rangle$

system to solve
$$\begin{cases} 4xy = \lambda 2x \\ 2x^2+6y^2+6y = \lambda 2y \\ \boxed{x^2+y^2=1} \end{cases}$$

$\Rightarrow 4xy - \lambda 2x = 0 \Rightarrow 2x(2y-\lambda)=0$

$x=0$ or $2y=\lambda$

$0^2+y^2=1$

when $x=0$ $y=\pm 1$

$2x^2+6y^2+6y = 4y^2$

$\quad -4y^2 \qquad\quad -4y^2$

$f(0,1) =$

$f(0,-1) = 1$

$f\left(\frac{\sqrt{8}}{3}, -\frac{1}{3}\right) =$

$f\left(-\frac{\sqrt{8}}{3}, -\frac{1}{3}\right) =$

$f(0,0) = 0$

$$\frac{2x^2 + 2y^2 + 6y}{2} = \frac{0}{2}$$

$$x^2 + y^2 + 3y = 0$$

$$1 + 3y = 0 \Rightarrow y = -1/3 \Rightarrow$$

$$x^2 + (-1/3)^2 = 1$$

$$x^2 + 1/9 = 1$$

$$x^2 = 8/9 \Rightarrow x = \pm \frac{\sqrt{8}}{3}$$

Interior

$$\langle f_x, f_y \rangle = \langle 0, 0 \rangle$$

$$\nabla f = \langle 4xy,\ 2x^2 + 6y^2 + 6y \rangle$$

$$\begin{cases} 4xy = 0 \Rightarrow x = 0 \text{ or } y = 0 \\ 2x^2 + 6y^2 + 6y = 0 \end{cases}$$

and

$$2(0)^2 + 6y^2 + 6y = 0 \qquad (0,-1)$$

$$6y^2 + 6y = 0 \qquad (0,0)$$

$$6y(y+1) = 0 \qquad y = 0 \text{ or } y = -1$$

$$2x^2 + 6(0)^2 + 6(0) = 0$$

$$2x^2 = 0$$

$$x = 0 \qquad (0,0)$$

$$f(x,y) = 2x^2y + 2y^3 + 3y^2$$

$$f(0,1) = 0 + 2(1)^3 + 3(1)^2 = 5$$ max @ 0,1 of 5

$$f(0,-1) = 1$$

$$f\left(\tfrac{\sqrt{8}}{3}, -1/3\right) = 2\left(\tfrac{\sqrt{8}}{3}\right)^2(-1/3) + 2(-1/3)^3 + 3(-1/3)^2 =$$

$$= 2\left(\tfrac{8}{9}\right)(-1/3) + 2(-1/27) + 3(1/9)$$

$$= -16/27 - 2/27 + \cancel{1/3}\ 9/27 = -9/27 = -1/3$$

$f(-\frac{\sqrt{8}}{3}, -1/3) = 2(-\sqrt{8}/3)^2(-1/3) + 2(-1/3)^3 + 3(-1/3)^2 = -1/3$

$=$ min @ $(\pm\frac{\sqrt{8}}{3}, -1/3)$ of $-1/3$

$f(0,0) = 0$

$\nabla f = 0$ Lagrange

$\uparrow$ $\uparrow$

$x^2 + y^2 \leq 9$ interior boundary

$<$ $=$

$x^2 + y^2 = 9$ boundary

Fall 2019 #6

Let
$$f(x,y) = \begin{cases} \dfrac{1 - e^{x^2+y^2}}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Which of the following is/are true?

P. $\lim_{(x,y)\to(0,0)} f(x,y) = -1$ ✓

P

Q. f is continuous at (0,0) X

R. f is differentiable at (0,0)

in polar $r^2 = x^2 + y^2$ $\quad r^2 = (-2)^2 + (2)^2 = 8$

$$f(r,\theta) = \begin{cases} \dfrac{1-e^{r^2}}{r^2} & r \neq 0 \\ 0 & r = 0 \end{cases}$$

$$\lim_{r \to 0} \frac{1-e^{r^2}}{r^2} = \frac{1-e^0}{0} = \frac{0}{0}$$

$$L'H = \frac{-2\cancel{r}e^{r^2}}{2\cancel{r}} = -e^{r^2} = -1$$
@ 0

is f differentiable @ (a,b)
if $f_x(a,b)$ & $f_y(a,b)$ exist AND f is continuous @ (a,b)

Clairut's Thm: if f is continuous

$$f_{xy} = f_{yx} \quad \ddot\smile$$

Fall 2019

14. If $z = f(x,y)$ has continuous second-order partial derivatives and $\underline{x = r^2 + t^2}$ and $\underline{y = 2r}$, find $\dfrac{\partial^2 z}{\partial t^2}$.

$$\frac{\partial x}{\partial t} = 2t$$

$$\frac{\partial y}{\partial t} = 0$$

Note: $\dfrac{\partial z}{\partial t} = (2t)\dfrac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial t}}_{2t} + \cancel{\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}}_{0}$$

$$\frac{\partial z}{\partial t} = 2t\left(\frac{\partial z}{\partial x}\right)$$

$$= 2\left(\frac{\partial z}{\partial x}\right) + 2t\left[\frac{d}{dt}\frac{\partial z}{\partial x}\right]$$

$z(x)$

$z'(x)$

$= x' z'(x)$

$$2t\left(\frac{\partial^2 z}{\partial x^2}\right)$$

T ↑

$$= 2\left(\frac{\partial z}{\partial x}\right) + (2t)^2 \frac{\partial^2 z}{\partial x^2} \qquad A$$

$$= 2\left(\frac{\partial z}{\partial x}\right) + 4t^2 \frac{\partial^2 z}{\partial x^2}$$

2018

#12 Evaluate the limit:

$$\lim_{(x,y)\to(1,2)} \frac{xy^2 - 4x}{y-2} = \frac{x(y^2-4)}{y-2}$$

$$= \frac{x\cancel{(y-2)}(y+2)}{\cancel{y-2}}$$

$$= x(y+2)$$

C $\quad$ @ $(1,2) = 1(2+2) = 4$