Homework #10 – **Higher-order derivatives**

Exercise 1. Show that $(x^n)^{(n)} = n!$, for all $n \in \mathbb{N}$.

Exercise 2. Give an example of a function that is three times differentiable but not three times continuously differentiable.

Exercise 3. Show that the Cauchy function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is infinitely differentiable. Verify that every Taylor polynomial of f centered about 0 is identically zero.

Exercise 4. Give an example of an infinitely differentiable function f on $(0, +\infty)$, such that $\lim_{x\to+\infty} f(x) = 0$ but $\lim_{x\to+\infty} f'(x) \neq 0$.

Exercise 5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathbb{R}$. Prove that if $f''(x) = 0$, then there exist $a, b \in \mathbb{R}$ such that $f(x) = ax + b$.

Exercise 6. Write Taylor's formula for $f(x) = e^x$ centered about 0.

Exercise 7. Find the n-th Taylor polynomial for the given functions

1. $f(x) = \frac{1}{1-x}$, centered about 0.
2. $f(x) = \sin(x)$, centered about 0.
3. $f(x) = \cos(x)$, centered about 0.
4. $f(x) = \ln(x)$, centered about 1.

Exercise 8. Use Taylor's theorem to prove that $1 - \frac{x^2}{2} \leq \cos(x)$, for all $x \in \mathbb{R}$.

Exercise 9. Let $f\colon (a, b) \to \mathbb{R}$ be twice differentiable. Let $c \in (a, b)$ be a critical point, that is $f'(c) = 0$.

1. Prove that if $f''(c) > 0$, then c is a local minimum of f.
2. Prove that if $f''(c) < 0$, then c is a local maximum of f.
3. Show that if $f''(c) = 0$, then c may or may not be a local extremum of f.