Homework #4 – **Cauchy sequences, subsequences**

Exercise 1. Find the set of all accumulation points for the given set S.

1. $S = \{x : 0 < |x - 1| < 3\}$.
2. $S = (-\infty, 2)$.
3. $S = \{x : x \in [0, 1] \text{ and } x \text{ is rational}\}$.

Exercise 2. Prove that a real number s_0 is an accumulation point of a set S if and only if there exists a sequence $\{a_n\}$ in S such that for every $n \in \mathbb{N}$, $a_n \neq s_0$ and $\lim_n a_n = s_0$.

Exercise 3. Determine which of the following sequences $\{a_n\}$ are Cauchy. Explain.

1. $a_n = \frac{1}{n}$
2. $a_n = \frac{n+1}{n}$
3. $a_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$
4. $a_n = \sum_{k=1}^{n} \frac{1}{k}$
5. $a_n = \sum_{k=1}^{n} \frac{(-1)^k}{k!}$
6. $|a_{n+1} - a_n| < r^n$, for some $r < 1$.

Exercise 4. Prove that if $\{a_n\}$ is a Cauchy sequence, then $\lim_{n\to+\infty} |a_{n+1} - a_n| = 0$. Is the converse true?

Exercise 5. Prove that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, so is $\{a_n + b_n\}$ and $\{a_n b_n\}$.

Exercise 6. Determine whether the sequence $\{b_n\}$ is a subsequence of $\{a_n\}$, with a_n and b_n given as

1. $b_n = -1$ and $a_n = (-1)^n$
2. $b_n = \frac{1}{\sqrt{n}}$ and $a_n = \frac{1}{n}$
3. $b_n = \frac{1}{3n^2}$ and $a_n = \frac{1}{n^2}$

Exercise 7. Verify that the following sequences diverge, and find all the subsequential limits.

1. $a_n = \frac{1+(-1)^{n+1}}{2}$
2. $a_n = \sin\left(\frac{n\pi}{2}\right)$
3. $a_n = (-1)^n \left(\frac{n-1}{n}\right)$

Exercise 8. Prove that every unbounded sequence contains a monotone subsequence that diverges to infinity.