Homework #5 – **Limits of functions, sided limits**

Exercise 1. Determine whether or not the given limits exist. Prove your assertion in each case, by using the definition or theorems.

1. $\lim_{x\to+\infty} \frac{-x^2}{2x+3}$.
2. $\lim_{x\to+\infty} \frac{\sqrt{x}}{x^2-1}$.
3. $\lim_{x\to+\infty} \frac{\sin(x)}{x}$.
4. $\lim_{x\to+\infty}(-1)^x$.
5. $\lim_{x\to+\infty} x\cos(x)$.

Exercise 2.

1. Verify that for all $x>0$, $\frac{x-1}{x} < \frac{\lfloor x\rfloor}{x} \leq 1$.
2. Evaluate $\lim_{x\to+\infty} \frac{\lfloor x\rfloor}{x}$.
3. Evaluate $\lim_{x\to-\infty} \frac{\lfloor x\rfloor}{x}$.

Exercise 3. Prove that if the limit of a function at a point exists, then the limit is unique.

Exercise 4. Prove that $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Exercise 5. Prove that $\lim_{x\to 0} \frac{1}{\sin(x)}$ does not exist.

Exercise 6. Evaluate the given limits and prove your conclusion using only the definition.

1. $\lim_{x\to 0} \frac{x^2}{|x|}$.
2. $\lim_{x\to 0}(x+1)^3$.
3. $\lim_{x\to 2} \frac{x^2+4}{x+2}$.
4. $\lim_{x\to 1} \frac{x^2-x-2}{2x-3}$.

Exercise 7. Use mathematical induction to prove that for all $n\in\mathbb{N}$, $\lim_{x\to a} x^n = a^n$.

Exercise 8.

1. Determine what is wrong with the following argument:

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\sin\left(\frac{1}{x}\right)\right) = 0\cdot(\text{anything}) = 0.$$

2. Use a correct method to find $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$.

Exercise 9. Use the definition to prove that $\lim_{x\to 1^+} \frac{x}{x-1} = +\infty$.

Exercise 10. Evaluate the given limits. Prove your assertion.

1. $\lim_{x\to -1^-} \lfloor x \rfloor$.
2. $\lim_{x\to -1^+} \lfloor x \rfloor$.
3. $\lim_{x\to 0^+} \exp\left(-\frac{1}{x}\right)$.
4. $\lim_{x\to 0^+} \frac{1}{x}$.
5. $\lim_{x\to 0^+} \cos\left(\frac{1}{x}\right)$.

Exercise 11. Prove the following theorem:
Let $a > 0$. Let $f : (0, a) \to \mathbb{R}$ be a function. If either of the limits

$$\lim_{x\to 0^+} f(x) \quad \text{or} \quad \lim_{x\to +\infty} f\left(\frac{1}{x}\right)$$

exist, then both limits exist and they are equal.