

Homework #7 – Uniform continuity

Exercise 1. Determine whether or not the given functions are uniformly continuous.

1. $f(x) = \frac{1}{x}$, with $x \in (0, 1]$
2. $f(x) = x^3$, with $x \in [0, 2]$
3. $f(x) = \frac{x}{x+4}$, with $x \in [0, 2]$
4. $f(x) = \sin\left(\frac{1}{x}\right)$, with $x \in (0, 1]$
5. $f(x) = \frac{1}{x^2}$, with $x \in [1, +\infty)$

Exercise 2. If f and g are uniformly continuous on D , prove that:

1. $f + g$ is uniformly continuous on D
2. cf is uniformly continuous on D
3. if f and g are both bounded on D , then fg is uniformly continuous on D
4. If D is bounded, then f is bounded

Exercise 3. Prove or find a counterexample to the following statements.

1. f bounded and continuous on D implies that f is uniformly continuous on D .
2. f uniformly continuous on D implies that f is bounded on D .
3. f continuous on $[a, b]$ implies that f is uniformly continuous on $[a, b]$.
4. f bounded on D implies that f is uniformly continuous on D .
5. f uniformly continuous on (a, b) implies that f is bounded on (a, b) .

Exercise 4. Give an example of a function f_1, f_2, f_3, f_4 that satisfies each of the given conditions.

1. f_1 continuous but not uniformly continuous on $[0, +\infty)$ with $\lim_{x \rightarrow +\infty} f_1(x) = -\infty$.
2. f_2 continuous but not uniformly continuous on $[0, +\infty)$ where $\lim_{x \rightarrow +\infty} f_2(x)$ does not exist.
3. f_3 uniformly continuous on $[0, +\infty)$ with $\lim_{x \rightarrow +\infty} f_3(x) = -\infty$.
4. f_4 uniformly continuous on $[0, +\infty)$ where $\lim_{x \rightarrow +\infty} f_4(x)$ does not exist.

Exercise 5. Give an example of a function f with a Lipschitz constant $L \geq 1$, such that

1. f has no fixed points
2. f has exactly one fixed point
3. f has infinitely many fixed points