Homework #8 – **Derivative of a function**

Exercise 1. Let f be defined on an interval (a, b), and define

$$g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

1. Prove that if f is differentiable at $c \in (a, b)$, then $f'(c) = g(c)$.
2. Find a function f and a point c such that $g(c)$ exists but f is not differentiable at c.

Exercise 2. For each given function f in their respective domain, find f' using the definition, if possible:

1. $f(x) = c$, c is a constant
2. $f(x) = \sqrt{x}$
3. $f(x) = x^{3/2}$
4. $f(x) = x^{-1/2}$
5. $f(x) = 2x^2 - x + 1$

Exercise 3. Determine if each function f is differentiable at the point indicated. If it is, find its derivative at that point; if not, explain why not.

1. $$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{at } x = 0$$
2. $$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{at } x = 0$$
3. $$f(x) = \begin{cases} x^2 + 2 & \text{if } x \leq 1 \\ 3x & \text{if } x > 1 \end{cases} \quad \text{at } x = 1$$
4. $$f(x) = \begin{cases} x^3 + 2x + 1 & \text{if } x \geq 2 \\ 4x + 1 & \text{if } x < 2 \end{cases} \quad \text{at } x = 2$$

Exercise 4. Give an example, if possible, of a function that is defined on $\mathbb{R}$ and is:

1. Continuous at exactly one point and differentiable at exactly one point.
2. Continuous at exactly one point and nowhere differentiable.
3. Differentiable at exactly two points and continuous at exactly one point.

Exercise 5. Prove that if f is differentiable, then for all $n \in \mathbb{N} \setminus \{0\}$, $(f^n)'(x) = n f^{n-1}(x) f'(x)$.