University of Florida **MAA 4102/5104**

Homework #9 – **Properties of differentiable function, Mean value theorems**

Exercise 1.

1. Prove that if f is differentiable and even, then f' is odd.
2. Prove that if f is differentiable and odd, then f' is even.

Exercise 2. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) except at $c \in (a, b)$. Prove that if $\lim_{x\to c} f'(x) = L$, then f is differentiable at c and $f'(c) = L$.

Exercise 3. If a function f is differentiable and c is a real constant, prove that $(cf)'(x) = cf'(x)$.

Exercise 4. Give an example of a function f that is differentiable on (a, b) but f' is not continuous on (a, b).

Exercise 5. Give an example of a function f that is differentiable at a such that $f'(a) \neq 0$, but yet f attains a relative extremum at a.

Exercise 6. Give an example of a function f that has an absolute minimum at 0, and f' alternates sign in any neighborhood of 0.

Exercise 7. Give an example of a function defined on (a, b) that is not continuous on (a, b) but attains both relative and absolute extrema.

Exercise 8. Evaluate $\arctan'(x)$.

Exercise 9. Give three functions for which exactly two out of three assumptions of Rolle's theorem are satisfied but for which the conclusion does not follow.

Exercise 10. Consider the function $f\colon [0, 1] \to \mathbb{R}$, defined by $f(x) = x^3 - x$. What does Rolle's theorem guarantee? Explain.

Exercise 11.

1. Suppose that the function f is differentiable on (a, b). Prove that $f'(x) \geq 0$ on (a, b) if and only if f is increasing on (a, b).
2. Is it true that if f is differentiable and strictly increasing on (a, b), then $f'(x) > 0$ on (a, b)? Explain.

Exercise 12.

1. Suppose that the function f is differentiable on (a, b) and f' is bounded on (a, b). Prove that f is uniformly continuous.
2. Give an example of a function f that is differentiable, uniformly continuous on (a, b), but f' is not bounded.

Exercise 13. Prove the given inequalities using the mean value theorem:

1. $1 + rx \leq (1 + x)^r$, for all $x, r \geq 0$.
2. $1 + x \leq e^x$, for all $x \in \mathbb{R}$.
3. $1 - x \leq e^{-x}$, for all $x \in \mathbb{R}$.
4. $|\sin(x) - \sin(y)| \leq |x - y|$, for all $x, y \in \mathbb{R}$. In particular, $\sin(x)$ is 1-Lipschitz.
5. $\cos(x) \geq 1 - \frac{x^2}{2}$, for all $x \geq 0$.