Homework #1 – **Review of Probability**

- **You must master the 1st semester part of this course – MAP 6472.**

Exercise 1.

Let $N\colon (\Omega,\mathcal{F}) \to (\mathbb{N},\mathcal{P}(\mathbb{N}))$ be a random variable with values in $\mathbb{N}$, and let $\{X_n\}_{n\geq 1}\colon (\Omega,\mathcal{F}) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$ be a sequence of random variables.

Show that X_N and $\sum_{k=1}^{N} X_k$ are random variables on $(\Omega,\mathcal{F})$.

Exercise 2.

Let $A, B \subset \Omega$. Recall that $\sigma(\{A\})$ is the smallest σ-algebra on Ω containing A.

1. Describe $\sigma(\{A\})$ and $\sigma(\{A\}) \cup \sigma(\{B\})$.
2. Is $\sigma(\{A\}) \cup \sigma(\{B\})$ a σ-algebra in general? What about $\sigma(\{A\}) \cap \sigma(\{B\})$?

Exercise 3.

Definition: Let $X\colon (\Omega,\mathcal{F}) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$ be a r.v. We denote by $\sigma(X)$ the smallest σ-algebra included in $\mathcal{P}(\Omega)$ such that X is $(\Omega,\sigma(X))$-measurable.

1. Show that $\sigma(X) = X^{-1}(\mathcal{B}(\mathbb{R}))$.
2. Y is $\sigma(X)$-measurable if and only if there exists a measurable function $g\colon (\mathbb{R},\mathcal{B}(\mathbb{R})) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$ such that $Y = g(X)$.
3. Show that
$$\sigma(X,Y) = \sigma(\sigma(X)\cup\sigma(Y)) = \sigma(X^{-1}(\mathcal{B}(\mathbb{R})) \cup Y^{-1}(\mathcal{B}(\mathbb{R}))).$$

Exercise 4.

Let $X, Y\colon (\Omega,\mathcal{F},\mathbb{P}) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$ be random variables such that $X \leq Y$ a.s.

1. Show that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.
2. Show that if $\mathbb{E}[X] = \mathbb{E}[Y]$, then $X = Y$ a.s.

Exercise 5.

Let $X\colon (\Omega,\mathcal{F},\mathbb{P}) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$ be a random variable. Show that

$$\forall A \in \mathcal{F},\ \mathbb{E}[X 1_A] = 0 \quad \Longrightarrow \quad X = 0 \text{ a.s.}$$

Exercise 6.

Let $\{X_n\}_{n\geq 1}$ be a decreasing sequence of non-negative r.v. such that $X_1 \in L^1$ (that is $\mathbb{E}[|X_1|] < +\infty$). Show that $\{X_n\}$ converges to X a.s. for some random variable X, and

$$\lim_{n\to+\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

Is the statement still valid if $\mathbb{E}[|X_1|] = +\infty$?