University of Florida **MAP 6473**

Homework #3 – **Martingale**

Exercise 1. Let $\{X_n\}$ be a sequence of i.i.d. random variables such that $\mathbb{E}[X_1^2] < +\infty$. Take $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$, and denote $S_n = \sum_{k=1}^n X_k$, $S_0 = 0$.

1. Define $M_n = (S_n - n\mathbb{E}[X_1])^2 - n\text{Var}(X_1)$, $n \geq 1$. Prove that $\{M_n\}$ is an $\mathcal{F}_n$-martingale.
2. Assume that $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$. For $u \in \mathbb{R}$, define $M_n^u = \cosh(u)^{-n} e^{uS_n}$. Prove that $\{M_n^u\}$ is an $\mathcal{F}_n$-martingale.

Exercise 2. Let $\{X_n\}$ be an $\mathcal{F}_n$-sub-martingale (resp. super-martingale) such that all X_n have same distribution.

1. Prove that $\{X_n\}$ is an $\mathcal{F}_n$-martingale.
2. Deduce that for all $a \in \mathbb{R}$, $X_n \wedge a$ and $X_n \vee a$ are martingales.

Exercise 3. (Wald's identity)

Let $\{X_n\}$ be a sequence of i.i.d. random variables in L^1. Take $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$, $n \geq 1$, and denote $S_n = \sum_{k=1}^n X_k$, $S_0 = 0$.

1. Let τ be an $\mathcal{F}_n$-integrable stopping time. Prove that S_τ is integrable, and
$$\mathbb{E}[S_\tau] = \mathbb{E}[X_1]\mathbb{E}[\tau].$$
2. Prove that if $\mathbb{E}[X_1] \neq 0$ and if τ is an $\mathcal{F}_n$-stopping time such that $\sup_n |\mathbb{E}[S_{n\wedge\tau}]| < +\infty$, then τ is integrable.

Exercise 4.
Let $\{X_n\}$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$. Denote $S_n = \sum_{k=1}^n X_k$, $S_0 = 0$, and $T = \inf\{n \geq 0 : S_n = 1\}$. Prove that $\mathbb{E}[T] = +\infty$.

Hint: One may use the Wald identity.

Exercise 5. (Doob's inequality)

Let $\{X_n\}$ be a non-negative $\{\mathcal{F}_n\}$-martingale. Denote $\tau = \inf\{n \geq 0 : X_n \geq c\}$, where $c > 0$.

1. Prove that τ is an $\{\mathcal{F}_n\}$-stopping time.
2. Prove that for all $j \leq n$, $\mathbb{E}[X_j 1_{\tau=j}] = \mathbb{E}[X_n 1_{\tau=j}]$.
3. Deduce that $\mathbb{E}[X_\tau 1_{\tau\leq n}] = \mathbb{E}[X_n 1_{\tau\leq n}]$, and that $c\mathbb{P}(\tau \leq n) \leq \mathbb{E}[X_n 1_{\tau\leq n}]$.
4. Prove that $\mathbb{P}(\sup_{k\leq n} X_k \geq c) \leq \frac{\mathbb{E}[X_n]}{c}$.

Exercise 6.

Let $\{X_n\}_{n\geq 0}$ be a sequence of random variables in L^1, adapted to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$. Prove that $\{X_n\}_{n\geq 0}$ is an $\{\mathcal{F}_n\}_{n\geq 0}$-martingale if and only if for all $\{\mathcal{F}_n\}_{n\geq 0}$-stopping time T bounded, one has $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Hint: One may consider for all $n \in \mathbb{N}$ and $B \in \mathcal{F}_n$, $T = n1_{B^c} + (n+1)1_B$.

Exercise 7. (The gambler)

In a favorable game, given $p \in (\frac{1}{2}, 1)$ and a sequence of i.i.d. random variables $\{\varepsilon_n\}$ such that

$$\mathbb{P}(\varepsilon_n = 1) = p, \qquad \mathbb{P}(\varepsilon_n = -1) = 1 - p,$$

where $\varepsilon_n = 1$ if the gambler wins at the n-th game and $\varepsilon_n = -1$ if he loses. The initial wealth of the gambler is $x_0 > 0$. Let X_n be his wealth at time n, and let C_n be the amount of money he bets at the $(n+1)$-th game. The gambler cannot ask for a loan.

The model is as follows: for each $n \in \mathbb{N}$, the gambler chooses C_n measurable with respect to $\mathcal{F}_n = \sigma\{\varepsilon_1, \ldots, \varepsilon_n\}$, with $0 \leq C_n \leq X_n$ and $X_{n+1} = X_n + C_n\varepsilon_{n+1}$.

1. (A risky strategy.) Here, we take $C_n = X_n$ for all $n \in \mathbb{N}$.
   (a) Prove that this strategy maximize $\mathbb{E}[X_n]$ for all n.
   (b) Define $T = \inf\{n \in \mathbb{N} : X_n = 0\}$. Determine the distribution of T, and deduce that $T < +\infty$ a.s.
   (c) Prove that $X_n = 0$ for all $n \geq T$, and that $\{X_n\}$ converges to 0 a.s.
2. (A cautious strategy.) Here, we take $C_n = \gamma_n X_n$, with $0 < \gamma_n < 1$.
   (a) For $n \geq 1$, define $M_n = \log(X_n) - \sum_{k=0}^{n-1} (p\log(1+\gamma_k) + (1-p)\log(1-\gamma_k))$ and $M_0 = \log(x_0)$. Prove that $\{M_n\}$ is an $\{\mathcal{F}_n\}$-martingale.
   (b) Prove that the choice $\gamma_n = 2p - 1$, for all n, maximize $\mathbb{E}[\log(X_n)]$.
   (c) For this strategy, that is $\gamma_n = 2p - 1$, compute the increasing process $< M >_n$ and study the limit in L^2 and almost sure of the sequence $\{\frac{\log(X_n)}{n}\}$.