UF | UNIVERSITY of FLORIDA
Department of Mathematics

CALCULUS III
FALL 2024
Version A

FINAL EXAM

A. Sign your bubble sheet on the back at the bottom in ink.

B. In pencil, write and encode in the spaces indicated:

1) Name (last name, first initial, middle initial)

2) UF ID number

3) **SKIP** Section number

C. Under "special codes" code in the test ID numbers 4, 1.

1 2 3 ● 5 6 7 8 9 0
● 2 3 4 5 6 7 8 9 0

D. At the top right of your answer sheet, for "Test Form Code", encode A.

● B C D E

E. **1)** This test consists of 18 multiple choice questions

2) The time allowed is 120 minutes.

3) You may write on the test.

4) Raise your hand if you need more scratch paper or if you have a problem with your test. DO NOT LEAVE YOUR SEAT UNLESS YOU ARE FINISHED WITH THE TEST.

F. KEEP YOUR BUBBLE SHEET COVERED AT ALL TIMES.

G. When you are finished:

1) Before turning in your test **check carefully for transcribing errors**. Any mistakes you leave in are there to stay.

2) You must turn in your scantron and tearoff sheets to your discussion leader or exam proctor. Be prepared to show your picture I.D. with a legible signature.

3) The answers will be posted in Canvas within one day after the exam.

Summary of Integration Formulas

- Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$ ✓

- Fundamental Theorem of Line Integrals ← conservative

$$\int_C \underbrace{\nabla f}_{= \vec{F}} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$ ✓

- Green's Theorem (circulation form) ✓

$$\iint_D \operatorname{curl} \vec{F} \cdot \hat{k}\, dA = \oint_C \vec{F} \cdot d\vec{r}$$

- Stokes' Theorem

$$\iint \operatorname{curl} F\, d\vec{S} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}$$

- Green's Theorem (flux form) 2D

$$\iint_D \operatorname{div} \vec{F}\, dA = \oint_C \vec{F} \cdot \hat{n}\, ds$$

- Divergence Theorem 3D

$$\iiint_E \operatorname{div} \vec{F}\, dV = \oiint_S \vec{F} \cdot \hat{n}\, dS$$

Questions 1 – 18 are worth 6 points each.

1. Let C be a simple, smooth, closed curve in the xy-plane, for how many of the following vector fields will the circulation $\oint_C \vec{F} \cdot d\vec{r}$ be equal to zero?

✓ i. $\vec{F}(x,y) = \langle 3, \pi^2 \rangle$ $\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$

✓ ii. $\vec{F}(x,y) = \langle 2y + x^2, \sin y + 2x \rangle$ $\frac{\partial P}{\partial y} = 2 = \frac{\partial Q}{\partial x}$

✓ iii. $\vec{F}(x,y) = \langle x + y, x - y \rangle$ $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}$

✗ iv. $\vec{F}(x,y) = \langle \cos y, \sin x \rangle$

$\frac{\partial P}{\partial y} = -\sin y \neq \frac{\partial Q}{\partial x}$

\# circulation of a conservative vector field is always zero

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

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2. Let C be a line segment with initial point $(1,2)$ and terminal point $(-3,5)$. Compute the line integral $\int_C 2x - y \, ds$.

$x_0 \; y_0 \qquad x_1 \; y_1$

$x = (1-t)(1) + t(-3) = 1 - t - 3t = 1 - 4t$

$y = (1-t)(2) + t(5) = 2 - 2t + 5t = 2 + 3t$

$r = \langle 1-4t, 2+3t \rangle$

$r' = \langle -4, 3 \rangle$

$|r'| = \sqrt{16+9} = 5$

$5\int_0^1 2(1-4t) - (2+3t)\, dt = 5\int_0^1 -11t \, dt = \frac{-55t^2}{2}\Big|_0^1 = \frac{-55}{2}$

(a) 0

(b) $-\frac{55}{2}$

(c) $-\frac{11}{2}$

(d) $-\frac{33}{2}$

(e) $-\frac{25}{2}$

3. If $\vec{F}(x,y,z) = \langle xe^y, z\ln x, xyz\rangle$, then which of the following vectors is parallel to $\nabla\times\vec{F}$ at the point $(1,0,-2)$.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ xe^y & z\ln(x) & xyz \end{vmatrix}$$

$$= \left\langle xz - \ln(x),\ -yz,\ \frac{z}{x} - xe^y \right\rangle$$

$$@\ (1,0,-2) = \langle -2, 0, -3\rangle$$

off by a scalar of -1 →

(a) $\langle 2,0,3\rangle$

(b) $\langle 2,1,0\rangle$

(c) $\langle 0,0,1\rangle$

(d) $\langle 2,-1,0\rangle$

(e) none of the above

a vector is parallel if it is off by a scalar: $c\langle -2, 0, -3\rangle$

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4. Evaluate $\int_C \overset{P}{(e^x y + x^2)}\, dx + \overset{Q}{(e^x + \cos(y))}\, dy$ where C is <u>any smooth curve</u> from $(1,0)$ to $(0,\pi)$.

$\frac{\partial P}{\partial y} = e^x = \frac{\partial Q}{\partial x} \Rightarrow F = \langle P, Q\rangle$ is conservative $\Rightarrow$ we can use the fundamental thm of line integrals

(a) $\frac{2}{3}$

(b) $\pi - \frac{1}{3}$

(c) 0

(d) π

(e) $-\pi$

$$\int P\,dx = \int e^x y + x^2\,dx = e^x y + \frac{1}{3}x^3 + C$$

$$\int Q\,dy = \int e^x + \cos(y)\,dy = ye^x + \sin(y) + C$$

$$f = ye^x + \frac{1}{3}x^3 + \sin(y) + C$$

$$f(0,\pi) - f(1,0) = (\pi) - \left(\frac{1}{3}\right) = \pi - \frac{1}{3}$$

5. Let S be the part of the plane $z = 4 + 2x + 6y$ defined over the region $\{(x, y) |\ 0 \le x \le 2,\ 0 \le y \le 1\}$ and oriented so that the z component of the normal vector is positive. If $\vec{F}(x, y, z) = \langle z, z, 12xy\rangle$, what is the flux of $\vec{F}$ across S?

If someone knows a shortcut way to do this one, let me know!

$$\text{Flux} = \iint_S F\cdot\hat{n}\, dS = \underbrace{\iint \vec{F}\cdot\langle -g_x, -g_y, 1\rangle\, dS}_{\text{when you have an explicit surface}}$$

plane: $z = 4 + 2x + 6y$

$n = \langle -g_x, -g_y, 1\rangle = \langle -2, -6, 1\rangle$

parameter: $r(x,y) = \langle x, y, 4 + 2x + 6y\rangle$

$F(r(x,y)) = \langle 4+2x+6y,\ 4+2x+6y,\ 12xy\rangle$

$F\cdot n = -8(4+2x+6y) + 12xy = -32 - 16x - 48y + 12xy$

$\int_0^1\int_0^2 -32-16x-48y+12xy\, dx\, dy = \int_0^1 -32x - 8x^2 - 48xy + 6x^2y\Big|_0^2 dy$

$= \int_0^1 -64-32-96y+24y\, dy = \int_0^1 -96-72y\, dy = -96y - 36y^2\Big|_0^1 = -132$

(a) 0

(b) -132

(c) -144

(d) -84

(e) -72

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6. Let S be the surface defined by $\vec{r}(u, v) = \langle 2u, -3v, v - 2u\rangle$ with domain $\{(u, v) |\ 0 \le u \le 1,\ -2 \le v \le 0\}$, then compute $\iint_S xy + 2z\, dS$.

(with $x = 2u$, $y = -3v$, $z = v - 2u$)

$dS = |\vec{n}| = |\vec{r}_u \times \vec{r}_v|$

$r_u = \langle 2, 0, -2\rangle \quad r_v = \langle 0, -3, 1\rangle$

$r_u \times r_v = \begin{vmatrix} i & j & k \\ 2 & 0 & -2 \\ 0 & -3 & 1\end{vmatrix} = \langle 6, -2, -6\rangle = \vec{n}$

$|\vec{n}| = \sqrt{36 + 4 + 36} = \sqrt{76}$

$\sqrt{76}\int_0^1\int_{-2}^0 -6uv + 2v - 4u\, dv\, du$

$= \sqrt{76}\int_0^1 -3uv^2 + v^2 - 4uv\Big|_{-2}^0 du$

$= \sqrt{76}\int_0^1 -(-12u + 4 + 8u)\, du$

$= \sqrt{76}\int_0^1 4u - 4\, du = \sqrt{76}(2u^2 - 4u)\Big|_0^1$

$= -2\sqrt{76}$

(a) $14\sqrt{76}$

(b) $6\sqrt{76}$

(c) $12\sqrt{76}$

(d) $20\sqrt{19}$

(e) none of the above

7. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = \langle ye^{xy}, xe^{xy} \rangle$ and if the curve C is $\vec{r}(t) = t\hat{i} - 8\hat{j}, 0 \le t \le 1$.

$r = \langle t, -8 \rangle$ $\quad F(r(t)) = \langle -8e^{-8t}, te^{-8t} \rangle$

$dr = \langle 1, 0 \rangle$

$\int_0^1 F \cdot dr\, dt = \int_0^1 -8e^{-8t}\, dt = e^{-8t} \Big|_0^1 = e^{-8} - 1$

(a) $e^{-8} - 2$

(b) 2

(c) 0

(d) $e^{-8} - 1$

(e) None of the above

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8. Evaluate $\int_C \sin(x)\, dx + z\cos(y)\, dy + \sin(y)\, dz$ where C is the ellipse $4x^2 + 9y^2 = 36$, oriented clockwise.

closed curve

$F = \langle \sin(x), z\cos(y), \sin(y) \rangle$

$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x} \quad \frac{\partial P}{\partial z} = 0 = \frac{\partial R}{\partial x} \quad \frac{\partial Q}{\partial z} = \cos(y) = \frac{\partial R}{\partial y}$

so F is a conservative vector field over a closed curve $\Rightarrow$ line integral is zero.

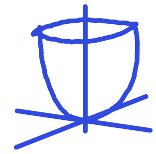
(a) 4

(b) 24

(c) 0

(d) 9

(e) 1



9. Find the surface area of the part of the surface $z = x^2 + y^2$ below the plane $z = 9$.

(a) $\frac{\pi}{6}(3\sqrt{3} - 1)$

(b) $\frac{\pi}{6}(3\sqrt{3} - 2\sqrt{2})$

(c) $\frac{\pi}{6}(37^{3/2} - 1)$

(d) $\frac{\pi}{6}(29^{3/2} - 1)$

(e) $\frac{\pi}{6}(27^{3/2} - 1)$

$$S.A. = \iint_A \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA$$

$$= \iint_A \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

use polar!

$9 = x^2 + y^2$ gives $0 \le r \le 3$

$$= \int_0^{2\pi}\int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$u = 1 + 4r^2 \quad du = 8r\,dr \quad \frac{du}{8r} = dr$

$$= \frac{1}{8}\int_0^{2\pi}\int_0^3 \sqrt{u}\, du\, d\theta = \frac{1}{8}\int_0^{2\pi} \frac{2}{3}u^{3/2}\Big|_0^3 d\theta =$$

$$\frac{1}{12}\int_0^{2\pi} (37)^{3/2} - 1\, d\theta = \frac{1}{12}(37^{3/2} - 1)\theta\Big|_0^{2\pi} = \frac{\pi}{6}(37^{3/2} - 1)$$

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10. Compute the tangent plane to the surface parametrized by $\vec{r} = u\hat{\imath} + uv\hat{\jmath} + (u + v)\hat{k}$ at the point (1,2,3).

(a) $3x + 2y + z = 10$

(b) $x - y + z = 2$

(c) $x + 2y + 3z = 14$

(d) $\langle x, y, z\rangle = \langle 1 + u, 2 + uv, 3 + u + v\rangle$

(e) $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$

$r = \langle u, uv, u+v \rangle$

$x = u \quad y = uv \quad z = u + v$

$1 = u \quad 2 = v \quad$ point $(u,v) = (1,2)$

$r_u = \langle 1, v, 1\rangle \quad r_v = \langle 0, u, 1\rangle$

$$\vec{n} = r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \langle 1, -1, 1\rangle$$

$$\langle 1, -1, 1\rangle \cdot \langle x-1, y-2, z-3\rangle = 0$$

$$x - 1 + 2 - y + z - 3 = 0$$

$$x - y + z = 2$$

P Q R

11. Let $\vec{F}(x,y,z) = \langle 2y^3, 1, e^z \rangle$. Use Stokes' Theorem to set up an integral to find the circulation of $\vec{F}$ along C, where C is the curve of intersection of the plane $y + 2z = 3$ and the cylinder $x^2 + y^2 = 4$. Orient C to be counterclockwise when viewed from above.

use polar

plane $y + 2z = 3$

$\vec{n} = \langle 0, 1, 2 \rangle$

(make $z = 1$): $\langle 0, \frac{1}{2}, 1 \rangle = \vec{n}$

(a) $\int_0^{2\pi} \int_0^2 -12r^3 \sin^2\theta \, dr \, d\theta$

(b) $\int_0^{2\pi} \int_0^2 12r^2 \cos^2\theta \, dr \, d\theta$

$\operatorname{curl} F = \nabla \times F = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$

(c) $\int_0^{2\pi} \int_0^2 12r^3 \cos^2\theta \, dr \, d\theta$

$\operatorname{curl} F = \langle 0-0, 0-0, 0-6y^2 \rangle = \langle 0, 0, -6y^2 \rangle$

(d) $\int_0^{2\pi} \int_0^2 -6r^2 \sin^2\theta \, dr \, d\theta$

$\langle 0, 0, -6y^2 \rangle \cdot \langle 0, \frac{1}{2}, 1 \rangle = -6y^2$

convert to polar

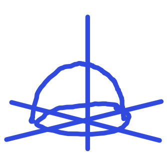
(e) $\int_0^{2\pi} \int_0^2 -6r^3 \sin^2\theta \, dr \, d\theta$

$\int_0^{2\pi} \int_0^2 -6r^2 \sin^2\theta \, r \, dr \, d\theta$

P Q R

12. Suppose $\vec{F}(x,y,z) = \langle xy^2, yz^2, zx^2 \rangle$ and let S be the union of the upper hemisphere of the sphere of radius 2 centered at the origin with the disc of radius 2 in the xy–plane centered at the origin such that S is positively oriented. Evaluate $\iint_S \vec{F}(x,y,z) \cdot d\vec{S}$. $= \iiint \operatorname{div} F \, dV$

since we have a closed region



$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = y^2 + z^2 + x^2 = \rho^2$ (spherical)

(a) 0

(b) $\frac{8\pi^2}{3}$

$\int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

(c) $\frac{16\pi^2}{3}$

$= \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{5}\rho^5 \sin\phi \Big|_0^2 \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/2} \frac{32}{5} \sin\phi \, d\phi \, d\theta$

(d) $\frac{64\pi}{5}$

$= \int_0^{2\pi} -\frac{32}{5} \cos\phi \Big|_0^{\pi/2} d\theta = \int_0^{2\pi} \frac{32}{5} \, d\theta = \frac{32}{5}\theta \Big|_0^{2\pi}$

(e) $\frac{128\pi}{5}$

$= \frac{64\pi}{5}$

13. Find the work done by the force field $\vec{F}(x, y) = \langle x, y + 2\rangle$ in moving an object along an arch of the cycloid $r(t) = \langle t - \sin t, 1 - \cos t\rangle$, $0 \le t \le 2\pi$.

$W = \int F \cdot dr$

$dr = \langle 1 - \cos t, \sin t \rangle$

(a) π^2

$F(r(t)) = \langle t - \sin t, 3 - \cos t \rangle$

(b) $2\pi^2$

$F \cdot dr = (t - \sin t)(1 - \cos t) + \sin t(3 - \cos t)$

$= t - t\cos t - \sin t + \cancel{\sin t \cos t} + 3\sin t - \cancel{\sin t \cos t}$

(c) $3\pi^2$

$= t - t\cos t + 2\sin t$

(d) $4\pi^2$

$\int_0^{2\pi} t - \underbrace{t\cos t}_{\text{tabular}} + 2\sin t \, dt = \frac{1}{2}t^2 + t\sin t + \cos t - 2\cos t \Big|_0^{2\pi}$

(e) None of the Above

$= 2\pi^2 + 1 - 2 - (1 - 2) = 2\pi^2$

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14. Let $f(x, y, z)$ be a potential function of the vector field $\vec{F} = \langle 2x, z\cos(yz), y\cos(yz)\rangle$. If $f(0, 0, 0) = -1$, find $f\left(2, 1, \frac{\pi}{2}\right)$

$\int 2x \, dx = x^2 + C$

(a) 1

$\int z\cos(yz) \, dy = \sin(yz) + C$

(b) 2

$\int y\cos(yz) \, dz = \sin(yz) + C$

(c) 3

$f = x^2 + \sin(yz) + C$

(d) 4

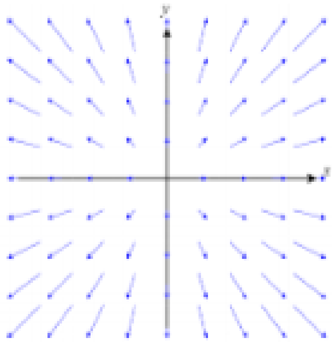
$f(0, 0, 0) = -1 = 0 + C \implies -1 = C$

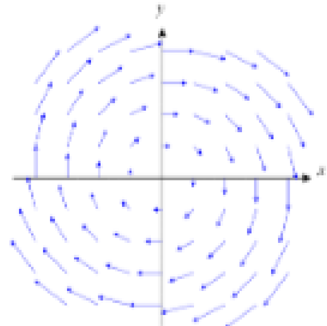
(e) 5

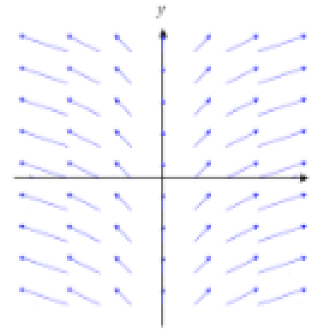
$f = x^2 + \sin(yz) - 1$

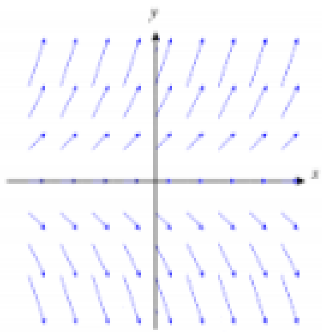
$f(2, 1, \frac{\pi}{2}) = 4 + \sin(\frac{\pi}{2}) - 1 = 4$

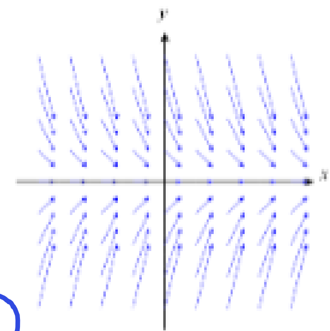
15. Sketch the gradient field of $f(x, y) = x - y^2$. $\nabla f = F = \langle 1, -2y \rangle$

a. 

b. 

c. 

d. 

e. 

x	y	F(x,y)
0	0	$\langle 1, 0 \rangle$
0	1	$\langle 1, -2 \rangle$
1	0	$\langle 1, 0 \rangle$
0	-1	$\langle 1, 2 \rangle$

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16. Let $\vec{F} = \langle \overset{P}{yz + 1}, \overset{Q}{xz + 1}, \overset{R}{xy + 1} \rangle$. Which of the following statements must be correct?

✓ P. $\vec{F}$ is conservative.

✗ Q. $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two smooth curves C_1 and C_2. } only if C_1, C_2 have same endpoints

✓ R. The circulation of $\vec{F}$ along a smooth curve is zero.
↳ circulation is always on a closed curve

a. P and Q only

b. P and R only

c. Q and R only

d. P, Q, and R

e. P only

$$\frac{\partial P}{\partial y} = z = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = y = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = x = \frac{\partial R}{\partial y}$$

F conservative ✓

17. Let $\vec{F} = \langle z - 2y, z + 2x, e^{-xy} \rangle$ and S be the part of the paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$ and oriented so that z component of the normal vector is positive. Calculate

$$\text{Flux} = \iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS = \int F \cdot dr$$

$r = \langle 3\cos t, 3\sin t, 0 \rangle$
$dr = \langle -3\sin t, 3\cos t, 0 \rangle$

$x = r\cos t$
$y = r\sin t$
$r = 3$

a. 9π

b. 18π

c. 36π

d. 48π

e. 0

$F(r(t)) = \langle -6\sin t, 6\cos t, e^{-9\cos t \sin t} \rangle$

$F \cdot dr = 18\sin^2 t + 18\cos^2 t = 18$

$\int_0^{2\pi} 18\, dt = 18t \Big|_0^{2\pi} = 36\pi$

you can do this one the long way with $\vec{n} = \langle 0,0,1 \rangle$

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18. You attend the University of

(a) Florida

(b) Missouri

(c) Arkansas

(d) Kansas