Answer the following problems. No calculators, formula sheets, or other aids are permitted. Please show all of your work. Simplify all solutions completely and clearly indicate your answers.

1. Find a power series representation for the following function. Be sure to state the radius of convergence and the interval of convergence.

$$f(x) = \frac{1}{2 - x^2}$$

algebraic simplification $f(x) = \frac{1}{2-x^2} = \frac{1}{2(1-\frac{x^2}{2})} = \frac{1}{2} \cdot \boxed{\frac{1}{1-\frac{x^2}{2}}} \rightarrow \frac{1}{1-x}$ ✓

formula: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

use formula and simplify $\frac{1}{2} \cdot \frac{1}{1-\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{1}{2} \cdot \left(\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x^{2n}}{2^n}\right) = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{2^{n+1}}}$

can see from this that the series will converge when

$\left|\frac{x^2}{2}\right| < 1 \Rightarrow \boxed{|x| < \sqrt{2}}$

OR use Root Test

$\lim_{n\to\infty} \left|\frac{x^{2n}}{2^{n+1}}\right|^{1/n} = \lim_{n\to\infty} \left|\frac{x^2}{2(2^{1/n})}\right| = \left|\frac{x^2}{2}\right| < 1$

$\frac{x^2}{2} < 1$ ✓ $\qquad \frac{x^2}{2} > -1$

$x^2 < 2 \qquad x^2 > -2$ X

$\boxed{|x| < \sqrt{2}}$

Thus, the interval of convergence is $(-\sqrt{2}, \sqrt{2})$. Neither endpoint is included because this is a geometric series.

2. Find the power series representation for the following function centered at $a = 0$ by differentiating a power series. Do NOT use Taylor's Theorem. You do not need to find the radius of convergence or the interval of convergence.

$$f(x) = \frac{2}{(1-2x)^2}$$ ← want to get rid of power

linear

integrate $\int f(x) = 2\int (1-2x)^{-2}\,dx = 2\left(\frac{1}{2}\cdot\frac{1}{1-2x}\right) = \frac{1}{1-2x} \rightarrow \frac{1}{1-x}$ ✓

$u = 1-2x$

$du = -2dx$

$\frac{du}{-2} = dx$

formula: $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

use formula $$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$

differentiate $$\frac{d}{dx}\cdot\frac{1}{1-2x} = \frac{d}{dx}\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} \frac{d}{dx}(2^n x^n)$$

re-index $$\frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} n\cdot 2^n\cdot x^{n-1} = \sum_{n=0}^{\infty} (n+1)2^{n+1}x^n$$

$$\sum_{n=0}^{\infty} n\cdot 2^n\cdot x^{n-1}$$ ← is also fine because it is equivalent to