THE LIMIT OF A FUNCTION, f(X):

THE LIMIT OF f(X) AS X APPROACHES a IS DENOTED $\lim_{x \to a} f(x)$. THIS LIMIT DETERMINES THE "VALUE" OF f(X) IN A NEIGHBORHOOD AROUND X.

ONE-SIDED LIMIT:

- $\lim_{x \to a^+} f(x)$ IS THE LIMIT OF f(X) AS X APPROACHES a FROM THE RIGHT
- $\lim_{x \to a^-} f(x)$ IS THE LIMIT OF f(X) AS X APPROACHES a FROM THE LEFT
- IF $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$, THEN $\lim_{x \to a} f(x)$ EXISTS
- IF $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, THEN $\lim_{x \to a} f(x)$ DOES NOT EXIST

HOW TO FIND LIMITS (WHEN X IS **NOT** APPROACHING $\pm\infty$):

- METHOD ONE: GRAPH THE FUNCTION AND FIND THE LIMIT BY ANALYZING THE GRAPH
- METHOD TWO: FIND LIMIT ALGEBRAICALLY

  STEP ONE: PLUG IN THE VALUE THAT X IS APPROACHING, IF YOU GET AN ANSWER, THEN THIS IS YOUR LIMIT

  STEP TWO: SIMPLIFY f(X) AND THEN PLUG IN THE VALUE THAT X IS APPROACHING TO THE SIMPLIFIED f(X), IF YOU GET AN ANSWER, THEN THIS IS YOUR LIMIT

  STEP THREE: WHEN YOU PLUG IN THE NUMBER THAT X IS APPROACHING, YOUR ANSWER WILL LOOK LIKE: $\frac{\text{NUMBER}}{0}$. I WILL

DISCUSS HOW TO SOLVE BELOW.

INFINITE LIMITS:

CONTINUING FROM STEP THREE ABOVE, IF YOU GET THE FORM $\frac{\text{"NUMBER"}}{0}$, THEN PLUG IN A NUMBER CLOSE TO THE NUMBER BEING APPROACHED. (I TYPICALLY ALWAYS PLUG IN A NUMBER WITHIN 0.1). THEN, ANALYZE THE SIGN.

EXAMPLE: $\lim_{x \to 2^+} \frac{x}{x-2} = \frac{2}{2-2} = \frac{2}{0}$ ↙ $\frac{\text{"NUMBER"}}{0}$

SINCE $x \to 2^{+}$ CHOOSE A NUMBER TO THE RIGHT OF 2, SAY 2.1. THEN, $\frac{x}{x-2} = \frac{2.1}{2.1-2} = \frac{\text{POSITIVE \#}}{\text{POSITIVE \#}} \Rightarrow \boxed{+\infty}$

$\lim_{x \to 2^+} \frac{x}{x-2} = +\infty$

EXAMPLE: $\lim_{x \to 2^-} \frac{x}{x-2} = \frac{2}{2-2} = \frac{2}{0}$ ↙ $\frac{\text{"NUMBER"}}{0}$

SINCE $x \to 2^{-}$ CHOOSE A NUMBER TO THE LEFT OF 2, SAY 1.9. THEN, $\frac{x}{x-2} = \frac{1.9}{1.9-2} = \frac{\text{POSITIVE \#}}{\text{NEGATIVE \#}}$ ↓ POSITIVE # DIVIDED BY A NEGATIVE # IS A NEGATIVE #, SO OUR ANSWER IS $-\infty$.

EXAMPLE: $\lim_{x \to 2} \frac{x}{x-2}$ DOES NOT EXIST BECAUSE

$\lim_{x \to 2^+} = +\infty$ AND $\lim_{x \to 2^-} = -\infty$,

SO $\lim_{x \to 2^+} \neq \lim_{x \to 2^-}$.

LIMITS TO REMEMBER:

- $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

- $\lim_{x \to 0} \ln(x) = -\infty$

LIMIT LAWS:

- $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

- $\lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

- $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

- $\lim_{x \to a} c f(x) = c \lim_{x \to a} f(x)$ (C IS A CONSTANT)

- $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$

- $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$

SQUEEZE THEOREM:

IF $f(x) \le g(x) \le h(x)$ AND IF $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, THEN ALSO

$\lim_{x \to a} g(x) = L$

HOW TO SOLVE:

1. START WITH A TRUE STATEMENT

2. MANIPULATE INEQUALITY UNTIL THE MIDDLE LOOKS LIKE YOUR ORIGINAL FUNCTION

3. TAKE THE LIMIT OF EACH PART OF THE INEQUALITY

EXAMPLE: $\lim_{x \to 0} x^3 \cos(x)$

1. $-1 \le \cos(x) \le 1$ (BECAUSE THE RANGE OF $\cos(x)$ IS $[-1,1]$.)
2. $-x^3 \le x^3 \cos(x) \le x^3$ (MULTIPLY BY x^3 TO GET ORIGINAL FUNCTION)
3. $\lim_{x \to 0} -x^3 \le \lim_{x \to 0} x^3 \cos(x) \le \lim_{x \to 0} x^3$

$$0 \le \lim_{x \to 0} x^3 \cos(x) \le 0$$

$$\Rightarrow \lim_{x \to 0} x^3 \cos(x) = 0$$

ABSOLUTE VALUE LIMITS: USE DEFINITION OF ABSOLUTE VALUE FUNCTION TO DETERMINE IF POSITIVE OR NEGATIVE.

EXAMPLE: $\lim_{x \to 3^+} \frac{|x-3|}{x^2-9}$

LOOK AT $|x-3|$: WHEN $x \to 3^+$, $x > 3$ (BECAUSE x IS APPROACHING 3 FROM THE RIGHT AND #S TO THE RIGHT OF 3 ARE BIGGER THAN 3), SO $|x-3| = x-3$

$$\Rightarrow \lim_{x \to 3^+} \frac{|x-3|}{x^2-9} = \lim_{x \to 3^+} \frac{\cancel{x-3}}{(x+3)\cancel{(x-3)}} = \lim_{x \to 3^+} \frac{1}{x+3} = \frac{1}{3+3} = \frac{1}{6}$$

EXAMPLE: $\lim_{x\to 3^-} \frac{|x-3|}{x^2-9}$

LOOK AT $|x-3|$: WHEN $x \to 3^-$, $x < 3$ (BECAUSE x IS APPROACHING 3 FROM THE LEFT, AND #S TO THE RIGHT OF 3 ARE SMALLER THAN 3), SO $|x-3| = -(x-3)$

$$\Rightarrow \lim_{x\to 3^-} \frac{|x-3|}{x^2-9} = \lim_{x\to 3^-} \frac{-\cancel{(x-3)}}{(x+3)\cancel{(x-3)}} = \lim_{x\to 3^-} \frac{-1}{x+3} = -\frac{1}{3+3} = -\frac{1}{6}$$

NOTE: $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$