XRONOS HW 18 #5:

CONSIDER $f(x)=x^2\ln(x)$. ON WHAT INTERVAL IS f INCREASING, DECREASING, CONCAVE UP, CONCAVE DOWN?

1. FIND CRITICAL POINTS:

$f'(x) = 2x\ln(x) + x^2\left(\frac{1}{x}\right) = 2x\ln(x) + x$

$f'(x)=0 \Rightarrow 2x\ln(x)+x=0 \Rightarrow x(2\ln(x)+1)=0$

SO, EITHER $x=0$ OR $2\ln(x)+1=0$

$2\ln(x)+1=0 \Rightarrow 2\ln(x)=-1 \Rightarrow \ln(x)=-1/2 \Rightarrow e^{\ln(x)}=e^{-1/2} \Rightarrow x=e^{-1/2}$

$x=0$ IS NOT IN THE DOMAIN OF f, SO $x=e^{-1/2}$ IS THE ONLY CRITICAL POINT

2. ANALYZE $f'(x)$ AROUND CRITICAL POINTS

(TEST POINTS (TRY $x=1$)

• WHEN $x>e^{-1/2}$, $x>0$ AND $2\ln(x)+1>0$, SO $f'(x)=x(2\ln(x)+1)>0$ → INCREASING

(TRY $x=1/8$

• WHEN $0<x<e^{-1/2}$, $x>0$ AND $2\ln(x)+1<0$, SO $f'(x)=x(2\ln(x)+1)<0$ → DECREASING

⇒ f IS INCREASING ON $(e^{-1/2}, \infty)$

f IS DECREASING ON $(0, e^{-1/2})$

3. FIND $f''(x)$:

$f'(x) = \underbrace{2x\ln(x)} + \underbrace{x} \Rightarrow f''(x) = \underbrace{2\ln(x) + 2x\left(\frac{1}{x}\right)} + \underbrace{1}$

$\Rightarrow f''(x) = 2\ln(x) + 3$

4. FIND x SUCH THAT $f''(x)>0$ AND $f''(x)<0$

$f''(x)=0 \Rightarrow 2\ln(x)+3=0 \Rightarrow \ln(x)=-3/2 \Rightarrow e^{\ln(x)}=e^{-3/2}$

$\Rightarrow x=e^{-3/2}$

• WHEN $x>e^{-3/2}$, $f''(x)=2\ln(x)+3>0$

• WHEN $0<x<e^{-3/2}$, $f''(x)=2\ln(x)+3<0$

SO, $\boxed{f \text{ IS CONCAVE UP ON } (e^{-3/2}, \infty)\text{; } f \text{ IS CONCAVE DOWN ON } (0, e^{-3/2})}$

5. LOCAL MIN AND MAX:

RECALL CRITICAL POINTS OF f (IN THE DOMAIN): $x = e^{-1/2}$

EVALUATE f'' AT $e^{-1/2}$:

$$f''(e^{-1/2}) = 2\ln(e^{-1/2}) + 3$$
$$= 2(-1/2) + 3$$
$$= -1 + 3 = 2 > 0$$

SINCE $f'(e^{-1/2}) = 0$, $f''(e^{-1/2}) > 0$, f HAS A LOCAL MIN AT $e^{-1/2}$:

$$f(e^{-1/2}) = (e^{-1/2})^2 \ln(e^{-1/2}) = e^{-1}(-1/2) = -1/2e$$

$\Rightarrow$ LOCAL MIN AT $\boxed{\left(e^{-1/2}, -\frac{1}{2e}\right)}$

6. INFLECTION POINTS:

$x = e^{-3/2}$: EVALUATE f AT $e^{-3/2}$:

$$f(e^{-3/2}) = (e^{-3/2})^2 \ln(e^{-3/2}) = e^{-3}\ln(e^{-3/2})$$

$\Rightarrow$ $\boxed{\text{INFLECTION POINT AT } (e^{-3/2}, e^{-3}\ln(e^{-3/2}))}$

<u>XRONOS HW 18 #4:</u>

(ON $[0, 2\pi]$)

CONSIDER $f(x) = \cos^2(x) - 2\sin(x)$. ON WHAT INTERVAL IS f INCREASING, DECREASING, CONCAVE UP, CONCAVE DOWN?

1. FIND CRITICAL POINTS:

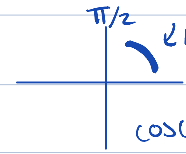
$$f'(x) = -2\cos(x)\sin(x) - 2\cos(x)$$
$$f'(x) = -2\cos(x)(\sin(x) + 1)$$
$$f'(x) = 0 \Rightarrow -2\cos(x)(\sin(x) + 1) = 0$$

SO, EITHER $-2\cos(x) = 0$ OR $\sin(x) + 1 = 0$

- $-2\cos(x) = 0 \Rightarrow \cos(x) = 0 \Rightarrow x = \pi/2, 3\pi/2$
- $\sin(x) + 1 = 0 \Rightarrow \sin(x) = -1 \Rightarrow x = 3\pi/2$

$x = \pi/2, 3\pi/2$ CRITICAL POINTS

2. ANALYZE $f'(x)$ AROUND CRITICAL POINTS ON $[0, 2\pi]$



← IN THE FIRST QUADRANT, ie $0 < x < \pi/2$, $\cos(x) > 0$, $\sin(x) > 0$

• WHEN $0 < x < \pi/2$, $-2\cos(x) < 0$ AND $\sin(x) + 1 > 0$

SO, $f'(x) < 0 \Rightarrow$ DECREASING

• WHEN $\pi/2 < x < 3\pi/2$, $\underline{-2\cos(x) > 0}$ AND $\underline{\sin(x) + 1 > 0}$

AGAIN, THINK ABOUT UNIT CIRCLE

SO, $f'(x) > 0 \Rightarrow$ INCREASING

• WHEN $3\pi/2 < x < 2\pi$, $-2\cos(x) < 0$ AND $\sin(x) + 1 > 0$, SO $f'(x) < 0 \Rightarrow$ DECREASING

$\Rightarrow$ $\boxed{\begin{array}{l} f \text{ IS INCREASING ON } (\pi/2, 3\pi/2) \\ f \text{ IS DECREASING ON } (0, \pi/2) \cup (3\pi/2, 2\pi) \end{array}}$

3. FIND $f''(x)$:

$f'(x) = \underline{-2\cos(x)\sin(x)} - \underline{2\cos(x)}$

$f''(x) = 2\sin(x)\sin(x) - 2\cos(x)\cos(x) + \underline{2\sin(x)}$

4. FIND X SUCH THAT $f''(x) > 0$ AND $f''(x) < 0$

$f''(x) = 2(\sin^2(x) - \cos^2(x) + \sin(x))$

$f''(x) = 2(\sin^2(x) - (1 - \sin^2(x)) + \sin(x))$

$f''(x) = 2(2\sin^2(x) - 1 + \sin(x))$

$f''(x) = 0 \Rightarrow 2(2\sin^2(x) - 1 + \sin(x)) = 0$

$\Rightarrow 2\sin^2(x) + \sin(x) - 1 = 0$

$\Rightarrow (2\sin(x) - 1)(\sin(x) + 1) = 0$

• $2\sin(x) - 1 = 0 \Rightarrow \sin(x) = 1/2 \Rightarrow x = \pi/6, 5\pi/6$

• $\sin(x) + 1 = 0 \Rightarrow \sin(x) = -1 \Rightarrow x = 3\pi/2$

SO, $f''(x) = 0$ WHEN $x = \pi/6, 5\pi/6, 3\pi/2$

• WHEN $0 < x < \pi/6$, $f''(x) < 0$

• WHEN $\pi/6 < x < 5\pi/6$, $f''(x) > 0$

• WHEN $5\pi/6 < x < 3\pi/2$, $f''(x) < 0$

• WHEN $3\pi/2 < x < 2\pi$, $f''(x) < 0$

$\Rightarrow$ $\boxed{\begin{array}{l} f \text{ IS CONCAVE UP ON } (\pi/6, 5\pi/6) \\ f \text{ IS CONCAVE DOWN ON } (0, \pi/6) \cup (5\pi/6, 3\pi/2) \cup (3\pi/2, 2\pi) \end{array}}$

5. LOCAL MIN AND MAX:

CRITICAL POINTS: $\pi/2$, $3\pi/2$

EVALUATE f'' AT CRITICAL POINTS:

• $f''(\pi/2) = 2(\sin^2(\pi/2) - \cos^2(\pi/2) + \sin(\pi/2)$

$= 2(1 - 0 + 1)$

$= 2(2) = 4 > 0$

• $f''(3\pi/2) = 2(\sin^2(3\pi/2) - \cos^2(3\pi/2) + \sin(3\pi/2)$

$= 2(-1 - 0 + (-1))$

$= 2(-2) = -4 < 0$

SINCE $f'(\pi/2) = 0$ AND $f''(\pi/2) > 0 \Rightarrow$ LOCAL MIN AT $\pi/2$

SINCE $f'(3\pi/2) = 0$ AND $f''(3\pi/2) < 0 \Rightarrow$ LOCAL MAX AT $3\pi/2$

EVALUATE f AT $\pi/2$, $3\pi/2$:

• $f(\pi/2) = \cos^2(\pi/2) - 2\sin(\pi/2) = 0 - 2 = -2$

• $f(3\pi/2) = \cos^2(3\pi/2) - 2\sin(3\pi/2) = 0 - 2(-1) = 0 + 2 = 2$

$\Rightarrow$ $\boxed{\text{LOCAL MIN AT } (\pi/2, -2), \text{ LOCAL MAX AT } (3\pi/2, 2)}$

6. INFLECTION POINTS:

$x = \pi/6$: $f(\pi/6) = \cos^2(\pi/6) - 2\sin(\pi/6) = \left(\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{1}{2}\right) = \frac{3}{4} - 1 = -\frac{1}{4}$

$x = 5\pi/6$: $f(5\pi/6) = \cos^2(5\pi/6) - 2\sin(5\pi/6) = \left(-\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{1}{2}\right) = \frac{3}{4} - 1 = -\frac{1}{4}$

$\Rightarrow$ $\boxed{\text{INFLECTION POINTS: } (\pi/6, -1/4), (5\pi/6, -1/4)}$

XRONOS HW 18 #1:

CONSIDER $f(x) = x^2 - x - \ln(x)$. ON WHAT INTERVAL IS f INCREASING, DECREASING, CONCAVE UP, CONCAVE DOWN?

1. FIND CRITICAL POINTS:

$$f'(x) = 2x - 1 - \frac{1}{x} = \frac{2x^2 - x - 1}{x}$$

$$f'(x) = 0 \Rightarrow \frac{2x^2 - x - 1}{x} = 0 \Rightarrow 2x^2 - x - 1 = 0 \Rightarrow (2x+1)(x-1) = 0$$

$x = -1/2, \ x = 1$

NOW, $f(x)$ IS NOT DEFINED AT $x = -1/2$, SO WE WILL ONLY CONSIDER $x = 1$:

2. ANALYZE $f'(x)$ AROUND $x = 1$:

IF $f'(x) > 0$, THEN f IS INCREASING, IF $f'(x) < 0$, THEN f IS DECREASING.

IF $x > 1$, THEN $(2x+1) > 0$ AND $(x-1) > 0$, SO $f'(x) = (2x+1)(x-1) > 0$

IF $0 < x < 1$, THEN $(2x+1) > 0$ AND $(x-1) < 0$, SO $f'(x) = (2x+1)(x-1) < 0$

$\Rightarrow$ f IS INCREASING ON $(1, \infty)$

$\Rightarrow$ f IS DECREASING ON $(0, 1)$

3. FIND $f''(x)$

$$f'(x) = 2x - 1 - x^{-1}, \text{ SO } f''(x) = 2 - (-1)x^{-2} = 2 + \frac{1}{x^2}$$

4. FIND x SUCH THAT $f''(x) > 0$ AND $f''(x) < 0$

OBSERVE THAT $f''(x) = 2 + \frac{1}{x^2}$ IS ALWAYS POSITIVE (NO MATTER WHAT x WE CHOOSE IN THE DOMAIN OF f, x^2 IS ALWAYS POSITIVE). SO,

f IS CONCAVE UP ON $(0, \infty)$

f IS NEVER CONCAVE DOWN

5. LOCAL MAX AND MIN:

RECALL THAT $f'(x) = 2x - 1 - \frac{1}{x}$ AND $f'(x)=0$ FOR $x=1$

$\uparrow$ ($x = -1/2$ NOT IN DOMAIN)

$f''(x) = 2 + \frac{1}{x^2} \Rightarrow f''(1) = 2 + \frac{1}{1} = 2+1 = 3 > 0$

$f'(1)=0$ AND $f''(1) = 3 > 0 \Rightarrow f$ HAS A LOCAL MIN @ $x=1$

SO, $f(1) = (1)^2 - 1 - \ln(1) = 1 - 1 - 0 = 0 \Rightarrow$ LOCAL MIN @ $(1,0)$

NO INFLECTION POINTS BECAUSE f DOES NOT CHANGE CONCAVITY.

XRONOS HW 18 #3:

CONSIDER $f(x) = x^4 e^{-x}$. ON WHAT INTERVAL IS f INCREASING, DECREASING, CONCAVE UP, CONCAVE DOWN?

1. FIND CRITICAL POINTS:

$f'(x) = 4x^3 e^{-x} + x^4(-e^{-x}) \Rightarrow f'(x) = 4x^3e^{-x} - x^4e^{-x}$

$f'(x)=0 \Rightarrow 4x^3e^{-x} - x^4e^{-x} = 0 \Rightarrow x^3e^{-x}(4-x) = 0$

SO, EITHER $x^3e^{-x}=0$ OR $4-x=0$

$x^3e^{-x} = 0 \Rightarrow x=0$

$4-x=0 \Rightarrow x=4$

2. ANALYZE $f'(x)$ AROUND CRITICAL POINTS

- WHEN $x<0$, $x^3e^{-x} < 0$, $4-x>0$, SO $f'(x) = x^3e^{-x}(4-x) < 0$ → DECREASING
- WHEN $0<x<4$, $x^3e^{-x}>0$, $4-x>0$, SO $f'(x) = x^3e^{-x}(4-x) > 0$ → INCREASING
- WHEN $x>4$, $x^3e^{-x}>0$, $4-x<0$, SO $f'(x)=x^3e^{-x}(4-x)<0$ → DECREASING

$\Rightarrow$ f IS INCREASING ON $(0,4)$

f IS DECREASING ON $(-\infty,0) \cup (4,\infty)$

3. FIND $f''(x)$:

$f'(x) = 4x^3e^{-x} - x^4e^{-x}$

$f''(x) = 12x^2e^{-x} - 4x^3e^{-x} - 4x^3e^{-x} + x^4e^{-x} = 12x^2e^{-x} - 8x^3e^{-x} + x^4e^{-x}$

4. FIND X SUCH THAT $f''(x) > 0$ AND $f''(x) < 0$

$f''(x) = x^2e^{-x}(12 - 8x + x^2)$

$f''(x) = 0 \Rightarrow x^2e^{-x}(x^2 - 8x + 12) = 0 \Rightarrow x^2e^{-x} = 0$ OR $x^2 - 8x + 12 = 0$

- $x^2e^{-x} = 0 \Rightarrow x = 0$
- $x^2 - 8x + 12 = 0 \Rightarrow (x-6)(x-2) = 0 \Rightarrow x = 6, 2$
- IF $x > 6$, $f''(x) > 0$
- IF $2 < x < 6$, $f''(x) < 0$
- IF $0 < x < 2$, $f''(x) > 0$
- IF $x < 0$, $f''(x) > 0$

$\boxed{f \text{ IS CONCAVE UP ON } (-\infty, 0) \cup (0, 2) \cup (6, \infty)}$

$\boxed{f \text{ IS CONCAVE DOWN ON } (2, 6)}$

5. LOCAL MAX AND MIN:

RECALL CRITICAL POINTS OF f, $x = 0, 4$.

$f''(x) = 12x^2e^{-x} - 8x^3e^{-x} + x^4e^{-x}$

$f''(0) = 0 - 0 + 0 = 0$ ← LOCAL MIN

$f''(4) = 12(4^2)e^{-4} - 8(4^3)e^{-4} + 4^4e^{-4} = e^{-4}(12 \cdot 16 - 8 \cdot 64 + 256)$

$= e^{-4}(-64) < 0$ ← LOCAL MAX

EVALUATE f AT THE CRITICAL POINTS $f(0) = 0^4e^{-0} = 0$

$f(4) = 4^4e^{-4} = \frac{4^4}{e^4}$

$\Rightarrow$ $\boxed{\begin{array}{l} \text{LOCAL MIN AT } x = 0, \ f(0) = 0 \\ \text{LOCAL MAX AT } x = 4, \ f(4) = \frac{4^4}{e^4} \end{array}}$

6. INFLECTION POINTS:

$x=2,\ x=6$

$x=2:\ f(2)=2^4e^{-2}=\frac{16}{e^2} \Rightarrow \boxed{\left(2, \frac{16}{e^2}\right)}$

$x=6:\ f(6)=6^4e^{-6}=\frac{6^4}{e^6} \Rightarrow \boxed{\left(6, \frac{6^4}{e^6}\right)}$

XRONOS HW 19 #6:

$$\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty} \frac{\sin(\pi/x)}{1/x}$$ THE LIMIT AS $x\to\infty$ OF $\sin(\pi/x)=0$ AND $1/x=0$.

APPLY L'HOPITAL'S RULE:

$$\frac{d}{dx}[\sin(\pi/x)] = \cos(\pi/x)\frac{d}{dx}(\pi x^{-1}) = \cos(\pi/x)(-\pi x^{-2}) = -\pi\cos(\pi/x)(1/x^2)$$

$$\frac{d}{dx}[1/x] = \frac{d}{dx}[x^{-1}] = -x^{-2} = -1/x^2.$$

$$\lim_{x\to\infty}\frac{\sin(\pi/x)}{1/x} = \lim_{x\to\infty}\frac{-\pi\cos(\pi/x)(1/x^2)}{-1/x^2}$$

$$= \lim_{x\to\infty}\frac{\cancel{+}\pi\cos(\pi/x)}{\cancel{x^2}}\cdot\left(+\frac{\cancel{x^2}}{1}\right)$$

$$= \lim_{x\to\infty}\pi\cos(\pi/x) = \boxed{\pi}$$

XRONOS HW 19 #7:

$$\lim_{x\to\infty}\sqrt{x}\,e^{-x/2} = \lim_{x\to\infty}\frac{\sqrt{x}}{e^{x/2}}$$

THE LIMIT A $x\to\infty$ OF $\sqrt{x}$, $e^{x/2}$ IS ∞, SO THIS IS THE ∞/∞ CASE.

APPLY L'HOPITAL'S RULE:

$$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}(x^{-1/2}) = \frac{1}{2\sqrt{x}}, \quad \frac{d}{dx}(e^{x/2}) = (e^{x/2})\left(\frac{1}{2}\right) = (e^{x/2})/2$$

$$\lim_{x\to\infty}\frac{\sqrt{x}}{e^{x/2}} = \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{\frac{e^{x/2}}{2}} = \lim_{x\to\infty}\frac{1}{2\sqrt{x}}\cdot\frac{2}{e^{x/2}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}\,e^{x/2}} = \boxed{0}$$

XRONOS HW 19 #8:

$$\lim_{x\to-\infty} x\ln\left(1-\frac{1}{x}\right) = \lim_{x\to-\infty}\frac{\ln(1-1/x)}{1/x}$$

THE LIMIT AS $x\to-\infty$ OF THE NUMERATOR AND DENOMINATOR ARE BOTH 0,

SO THIS IS THE $0/0$ CASE

APPLY L'HOPITAL'S RULE:

$$\frac{d}{dx}\left(\ln(1-1/x)\right) = \frac{1}{(1-1/x)}\left(\frac{d}{dx}(1-x^{-1})\right) = \frac{1}{1-1/x}\left(-(-x^{-2})\right) = \frac{1}{x^2(1-1/x)}$$

$$\frac{d}{dx}(1/x) = \frac{d}{dx}(x^{-1}) = -x^{-2} = -1/x^2$$

$$\lim_{x\to-\infty} \frac{\ln(1-1/x)}{1/x} = \lim_{x\to-\infty} \frac{\frac{1}{x^2(1-1/x)}}{-\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{1}{\cancel{x^2}(1-1/x)} \cdot \left(-\frac{\cancel{x^2}}{1}\right)$$

$$= \lim_{x\to-\infty} \frac{-1}{1-1/x} = \frac{-1}{1} = \boxed{-1}$$

XRONOS HW 19 #3:

$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}$ SINCE $\lim_{x\to\infty} \ln(x) = \infty$ AND $\lim_{x\to\infty} \sqrt{x} = \infty$, THIS IS THE $\frac{\infty}{\infty}$ CASE.

APPLY L'HOPITAL'S RULE:

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}, \quad \frac{d}{dx}[x^{1/2}] = \frac{1}{2\sqrt{x}}$$

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{x} \cdot 2(x^{1/2}) = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = \boxed{0}$$