Module 1 Lecture Notes

MAC1105

Summer B 2019

1 Real and Complex Numbers

1.1 Subgroups of Real Numbers

Definition

A set is a collection of MATHEMATICAL OBJECTS. An ELEMENT is an object that is in a specified set. An interval is a collection of REAL NUMBERS.

↑ WE WILL LEARN ABOUT REAL NUMBERS BELOW

Note 1. We can describe the elements of a set using **set builder notation**. An example of this is shown below.

Example 1. To describe the set of all freshman at the University of Florida in set builder notation, we would write:

$\{x : x \text{ IS A FRESHMAN AT UF}\}$

"THE SET OF ALL X SUCH THAT X IS A FRESHMAN AT UF"

OR, IN SET ROSTER NOTATION: {FRESHMAN 1, FRESHMAN 2, . . .}

Natural Numbers

The **natural numbers** are the numbers we use for counting:

$\{1, 2, 3, 4, \ldots\}$

1. THE NATURAL NUMBERS <u>DO NOT</u> INCLUDE 0
2. THE NATURAL NUMBERS <u>DO NOT</u> INCLUDE NEGATIVE NUMBERS
3. SYMBOL: $\mathbb{N}$

Whole Numbers

The set of natural numbers plus zero is the set of **whole numbers**:

$\{0, 1, 2, 3, 4, \ldots\}$

1. THE WHOLE NUMBERS <u>DO</u> INCLUDE 0
2. THE WHOLE NUMBERS <u>DO NOT</u> INCLUDE NEGATIVE NUMBERS
3. WHOLE NUMBERS = 0 + NATURAL NUMBERS
4. SYMBOL: $\mathbb{W}$

Integers

The set of **integers** is the set of negative natural numbers plus the whole numbers:

$$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

1. INTEGERS = NEGATIVE NATURAL NUMBERS + WHOLE NUMBERS

2. SYMBOL: $\mathbb{Z}$

Rational Numbers

The set of **rational numbers** includes fractions written as $\frac{m}{n}$, where m, n are INTEGERS and $n \neq 0$:

$$\{x : x = \frac{m}{n}, \text{ M AND N ARE INTEGERS AND } n \neq 0\}$$

1. SYMBOL: $\mathbb{Q}$
2. ANY INTEGER m CAN BE WRITTEN AS $\frac{m}{1}$, SO ALL INTEGERS ARE RATIONAL NUMBERS
3. RATIONAL NUMBERS ALSO INCLUDE REPEATING AND TERMINATING DECIMALS:

$0.3333\ldots = 0.\overline{3}$ IS A REPEATING DECIMAL

$0.365365365\ldots = 0.\overline{365}$ IS A REPEATING DECIMAL

$0.25 = \frac{1}{4}$ IS A TERMINATING DECIMAL (IT ENDS)

$0.1592439\ldots$ IS A NONTERMINATING DECIMAL (DOES NOT END)

Irrational Numbers

The set of **irrational numbers** is the set of numbers that are not RATIONAL, are non-REPEATING, and are non-TERMINATING.

1. IN SET-BUILDER NOTATION:

$$\{x : x \text{ IS A REAL NUMBER AND } x \text{ IS NOT A RATIONAL NUMBER}\}$$

2. IRRATIONAL NUMBERS INCLUDE ANY REAL NUMBER THAT IS NOT A RATIONAL NUMBER

3. EXAMPLES:

$\pi = 3.1415926\ldots$

NON-REPEATING & NON-TERMINATING

$\sqrt{2} = 1.41421\ldots$

NON-REPEATING & NON-TERMINATING

4. NON-EXAMPLES:

$\sqrt{4} = 2$

$0.21532153\ldots = 0.\overline{2153}$

REPEATING DECIMAL

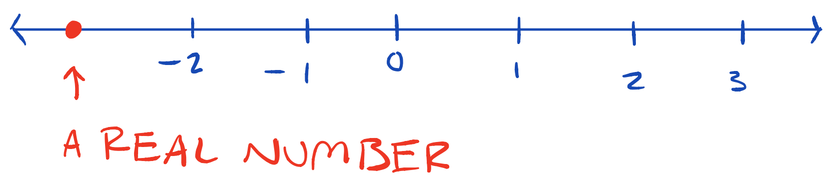
Real Numbers

The set of **real numbers** is the set of rational and irrational numbers together:

{X : X IS A RATIONAL OR IRRATIONAL NUMBER}

1. A NUMBER CANNOT BE BOTH RATIONAL AND IRRATIONAL

2. THE REAL NUMBERS CAN BE VISUALIZED ON A NUMBER LINE (THE REAL NUMBER LINE)



3. SYMBOL: $\mathbb{R}$

★ THE NUMBERS THAT ARE NOT REAL NUMBERS COMMONLY OCCUR WHEN WE TAKE THE SQUARE ROOT (OR THE EVEN ROOT) OF A NEGATIVE NUMBER AND WHEN WE DIVIDE BY 0 (YOU CANNOT DIVIDE BY 0!)

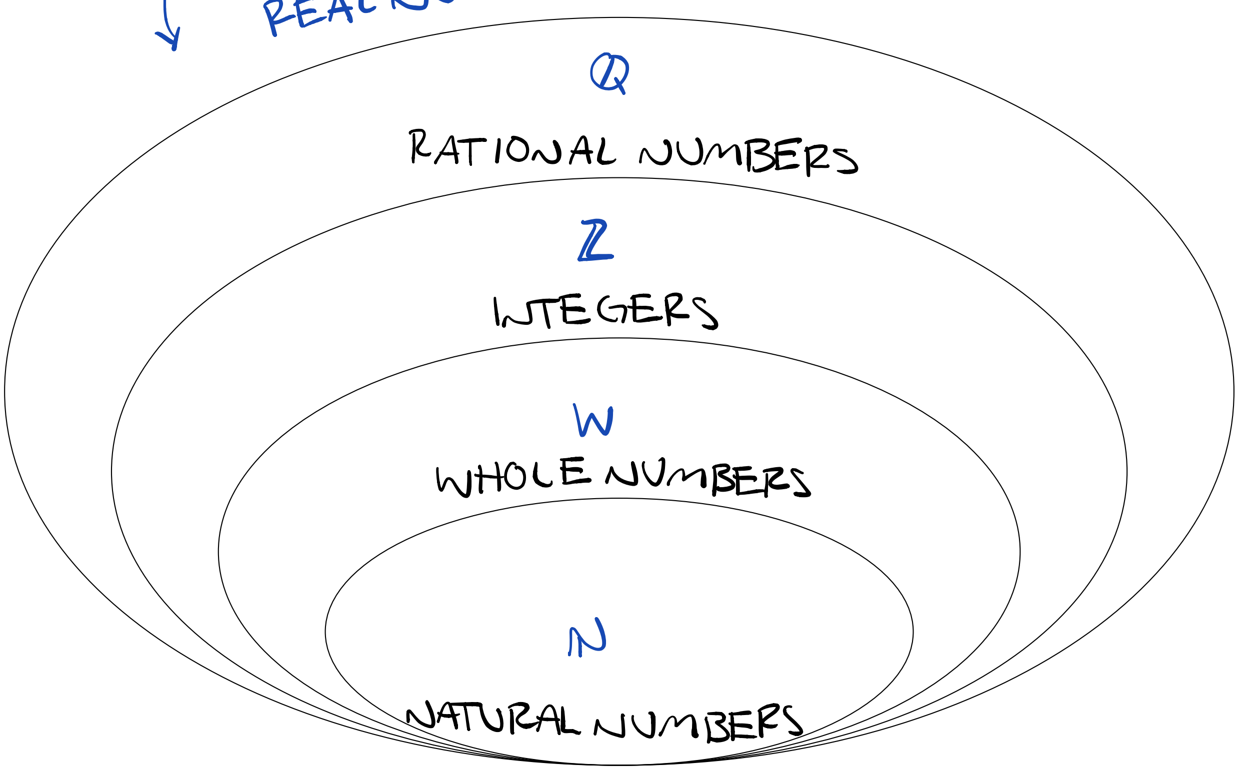
Definition

Let A and B be sets. Say that A is a SUBSET of B, written $A \subseteq B$, if every element of A is an element of B.

1. THE NATURAL NUMBERS ARE A SUBSET OF THE WHOLE NUMBERS BECAUSE EVERY NATURAL NUMBER IS ALSO A WHOLE NUMBER! $\{1, 2, 3, \ldots\} \subseteq \{0, 1, 2, 3, \ldots\}$

2. THE WHOLE NUMBERS ARE NOT A SUBSET OF THE NATURAL NUMBERS BECAUSE 0 IS NOT A NATURAL NUMBER

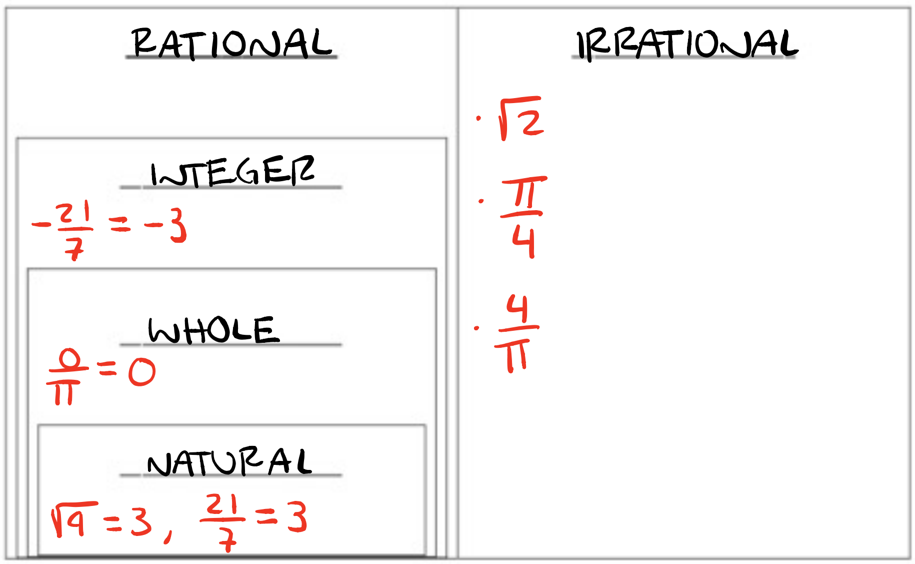
ALL SUBSETS OF THE REAL NUMBERS



Example 2. Classify the following numbers in the chart provided:

$$-\frac{21}{7}, \frac{21}{7}, \frac{\pi}{4}, \frac{4}{\pi}, \frac{0}{\pi}, \frac{\pi}{0}, \sqrt{9}, \sqrt{-9}, \sqrt{2}, \sqrt{-2}$$

$\sqrt{-2}$, $\sqrt{-9}$, $\frac{\pi}{0}$ ARE NOT REAL NUMBERS



Example 3. Which of the following is the smallest set of real numbers that $\sqrt{\frac{0}{36}}$ belongs to?

1. Natural Number
2. Whole Number
3. Integer
4. Rational
5. Irrational
6. Not a Real Number

$\frac{0}{36} = 0$ SO $\sqrt{\frac{0}{36}} = \sqrt{0} = 0$

NOT A NATURAL NUMBER, BUT IT IS A WHOLE NUMBER

Example 4. Which of the following is the smallest set of real numbers that 1.324324324324324... belongs to?

$1.324324\ldots = 1.\overline{324}$

1. Natural Number
2. Whole Number
3. Integer
4. Rational
5. Irrational
6. Not a Real Number

A REPEATING DECIMAL!

★ RECALL: RATIONAL NUMBERS INCLUDE REPEATING AND TERMINATING DECIMALS

1.2 Subgroups of Complex Numbers

Definition

A complex number is a number of the form $a + bi$, where a is the real part of the complex number and b is the IMAGINARY part of the complex number.

Definition

Let $a+bi$ be a complex number. If $a = 0$ and if $b \neq 0$, then $a+bi$ is called a PURE IMAGINARY number. An imaginary number is an even root of a negative integer.

1. $\sqrt{-1} = i$
2. $i^2 = -1$
3. $\underbrace{a}_{\text{REAL PART}} + \underbrace{bi}_{\text{IMAGINARY PART}}$
4. IF $b=0$ THEN $a+bi = a + 0i = a + 0 = a$ IS A REAL NUMBER
5. AN IMAGINARY NUMBER IS AN EVEN ROOT OF A NEGATIVE NUMBER
6. EVERY REAL NUMBER IS A COMPLEX NUMBER
7. EXAMPLES:

$$\sqrt{-2} = \sqrt{-1 \cdot 2} = \sqrt{-1}\,\sqrt{2} = i\sqrt{2}$$

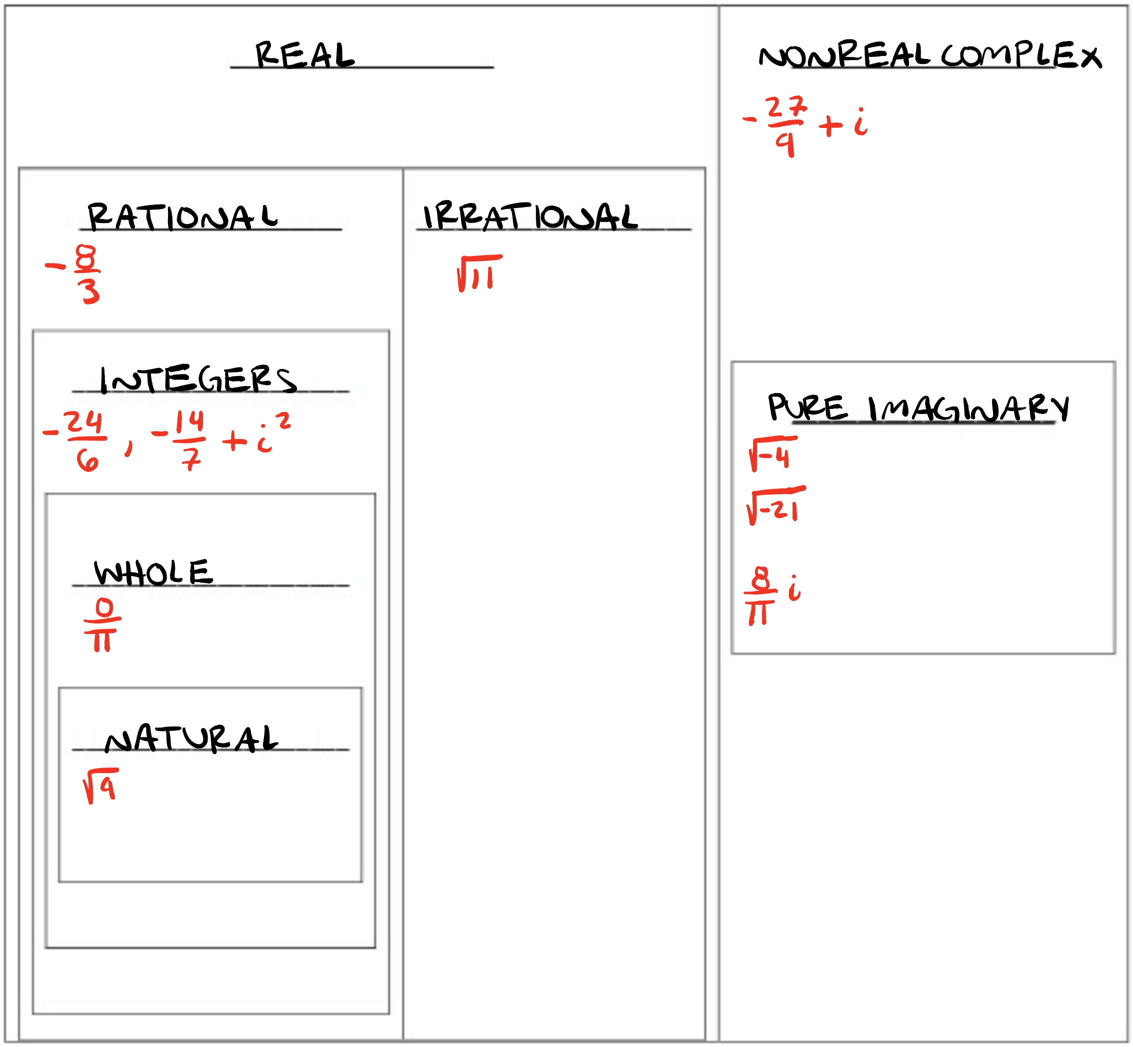
$$\sqrt{-4} = \sqrt{-1 \cdot 4} = \sqrt{-1}\,\sqrt{4} = i\sqrt{4} = i2 = 2i$$

★ DIVIDING BY 0 DOES NOT GIVE US A COMPLEX NUMBER!

$\frac{3}{0}$ IS NOT REAL AND NOT COMPLEX

Example 5. Classify the following numbers in the chart provided:

$$-\frac{27}{9}+i, \sqrt{-4}, \sqrt{-21}, \sqrt{9}, \sqrt{11}, \frac{\pi}{0}i, \frac{8}{\pi}i, -\frac{24}{6}, -\frac{8}{3}, \frac{0}{\pi}$$



★ $\frac{\pi}{0}i$ IS NOT A COMPLEX NUMBER!

1.3 Order of Operations

Properties of Real Numbers

Let a, b, and c be real numbers.

The Inverse Properties:

There is a unique number 0, called the **additive identity** such that:

1. $0 + a = a$

2. $a + 0 = a$

There is a unique number 1, called the **multiplicative identity** such that:

1. $a \cdot 1 = a$

2. $1 \cdot a = a$

The Identity Properties:

There is a unique number $-a$, called the **additive inverse** or **negative** of a such that:

1. $a + (-a) = a - a = 0$

2. $(-a) + a = -a + a = 0$

If $a \neq 0$, there is a unique number $\frac{1}{a}$, called the **multiplicative inverse** or **reciprocal** of a such that:

1. $a \cdot \left(\frac{1}{a}\right) = \frac{a}{a} = 1$

2. $\left(\frac{1}{a}\right) \cdot a = \frac{a}{a} = 1$

The Closure Property:

1. $a + b$ IS A REAL NUMBER

2. $a \cdot b$ IS A REAL NUMBER

WHEN YOU ADD AND MULTIPLY REAL NUMBERS, THE RESULT IS A REAL NUMBER

The Commutative Properties:

1. $a+b = b+a$ "ORDER DOES NOT MATTER"

2. $a \cdot b = b \cdot a$

The Associative Properties:

1. $a+(b+c) = (a+b)+c$

2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

The Distributive Property:

1. $a(b+c) = a \cdot b + a \cdot c$

2. $(a+b)c = a \cdot c + b \cdot c$

Definition

Operations in mathematics must be performed in a systematic order, which can be remembered by the acronym PEMDAS:

P - PARENTHESIS

E - EXPONENTS

M - MULTIPLICATION

D - DIVISION

A - ADDITION

S - SUBTRACTION

★ DIVISION IS MULTIPLICATION

★ SUBTRACTION IS ADDITION

$\Rightarrow$ "PEMA"

Note 2. In other words, to simplify mathematical expressions, we will:

1. Simplify any expressions within grouping symbols, such as () and []

2. Simplify any expressions containing EXPONENTS or RADICALS

3. Perform MULTIPLICATION and division IN THE ORDER IN WHICH THEY APPEAR from left to right

4. Perform addition and SUBTRACTION IN THE ORDER IN WHICH THEY APPEAR from left to right

Example 6. Use the order of operations (PEMDAS) to simplify the following expressions:

1. $12 - 10 \div (2 * 5)$ $= 12 - 10 \div 10$

   $10 \div 10 = 1$

   $= 12 - 1$

   $= 11$

2. $7 + 10^2 \div (2 * 5) + 12$ $= 7 + 10^2 \div 10 + 12$

   $10^2 = 100$

   $= 7 + 100 \div 10 + 12$

   $100 \div 10 = 10$

   $= 7 + 10 + 12$

   $= 17 + 12 = 29$

3. $3(2)^2 - 4(6 + 2)$ $= 3 \cdot 2^2 - 4 \cdot 8$

   $2^2 = 4$

   $= 3 \cdot 4 - 4 \cdot 8$

   $3 \cdot 4 = 12$

   $= 12 - 4 \cdot 8$

   $4 \cdot 8 = 32$

   $= 12 - 32$

   $= -20$

1.4 Operate on Complex Numbers

Note 3. $\sqrt{-a} = \sqrt{-1 \cdot a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$

Adding and Subtracting Complex Numbers

Adding Complex Numbers:

$$(a + bi) + (c + di) = \underline{(a+c)} + \underline{(b+d)}\, i$$

Subtracting Complex Numbers:

$$(a + bi) - (c + di) = \underline{(a-c)} + \underline{(b-d)}\, i$$

Multiplying Complex Numbers

Multiplying Complex Numbers by Real Numbers:

$$k(a + bi) = \underline{ka} + \underline{kb}\, i$$

Multiplying Complex Numbers by Complex Numbers: FOIL!
"FIRST, OUTER, INNER, LAST"

$$(a+bi)\cdot(c+di) = ac + adi + bci + bdi^2$$
$$= ac + (ad+bc)i + bd(-1)$$
$$= (ac-bd) + (ad+bc)i$$

Example 7. Multiply:

$$(3-4i)(2+3i)$$

$$= 3\cdot 2 + 3\cdot 3i + (-4i)\cdot 2 + (-4i)\cdot(3i)$$
$$= 6 + 9i - 8i - 12i^2$$
$$= 6 + (9-8)i - 12(-1)$$
$$= 6 + i + 12 = \boxed{18+i}$$

Example 8. Multiply:

$$(2+3i)(4-i)$$

$$= 2\cdot 4 + 2(-i) + 3i\cdot 4 + (3i)\cdot(-i)$$
$$= 8 - 2i + 12i - 3i^2$$
$$= 8 + (-2+12)i - 3(-1)$$
$$= 8 + 10i + 3 = \boxed{11+10i}$$

Definition

The complex conjugate of a complex number, $(a+bi)$ is $\underline{a-bi}$. In other words, the complex conjugate of a complex number is found by changing the sign of the imaginary part of the complex number.

Note 4. The product of $a+bi$ with its complex conjugate, $a-bi$ is $(a+bi)(a-bi) = a^2+b^2$.

$$(a+bi)(a-bi) = a\cdot a + a(-bi) + a(bi) + (bi)(-bi)$$
$$= a^2 - a\cancel{b}i + a\cancel{b}i - b^2i^2$$
$$= a^2 - b^2(-1) = a^2+b^2$$

EXAMPLES:

1. COMPLEX CONJUGATE OF $1+3i$ IS $1-3i$

2. COMPLEX CONJUGATE OF $-2i$ IS $-(-2i) = 2i$

Dividing Complex Numbers

To divide $a + bi$ by $c + di$, where c and d are both nonzero, multiply the fraction by the complex conjugate of $c + di$:

$$\left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right) = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac - adi + bci - bdi^2}{c^2+d^2}$$

$$= \frac{ac + (-ad+bc)i - bd(-1)}{c^2+d^2}$$

$$= \frac{(ac+bd) + (bc-ad)i}{c^2+d^2}$$

Example 9. Divide : $\dfrac{2+5i}{4-i}$

*COMPLEX CONJUGATE OF $4-i$ IS $4+i$:

$$\left(\frac{2+5i}{4-i}\right)\left(\frac{4+i}{4+i}\right) = \frac{(2+5i)(4+i)}{(4-i)(4+i)}$$

$$= \frac{2\cdot 4 + 2\cdot i + 5i\cdot 4 + 5i\cdot i}{4^2+1^2}$$

$$= \frac{8 + 2i + 20i + 5i^2}{16+1}$$

$$= \frac{8 + (2+20i) + 5(-1)}{17}$$

$$= \frac{8 + 22i - 5}{17} = \boxed{\frac{3+22i}{17} = \frac{3}{17} + \frac{22i}{17}}$$