Module 10L Lecture Notes

MAC1105

Summer B 2019

10 Synthetic Division

10.1 Divide With Synthetic Division

The Division Algorithm

The Division Algorithm states that, given a polynomial dividend, $f(x)$, and a nonzero polynomial divisor, $d(x)$, where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

$q(x)$ is the QUOTIENT and $r(x)$ is the REMAINDER. The remainder is either 0 or has degree strictly less than $d(x)$. If $r(x) = 0$, then $d(x)$ DIVIDES EVENLY into $f(x)$. This means that $d(x)$ and $q(x)$ are FACTORS of $f(x)$.

How to use Long Division to Divide a Polynomial by a Binomial

1. Set up the division problem.

2. Determine the first term of the quotient by dividing the leading term of the

DIVIDEND by the leading term of the DIVISOR.

3. Multiply the answer by the divisor and write it below the LIKE TERMS of the dividend.

4. Subtract the bottom BINOMIAL from the top binomial.

5. Bring down the next term of the dividend.

6. Repeat steps 2-5 until you reach the last term of the dividend.

7. If the remainder is non-zero, express the answer using the divisor as the DENOMINATOR.

Example 1. Use long division to divide $4x^3 + 12x^2 - 24x - 28$ by $x + 4$

$$
\begin{array}{r|l}
 & 4x^2 - 4x - 8 \\
\hline
x+4 & 4x^3 + 12x^2 - 24x - 28 \\
 & -(4x^3 + 16x^2) \downarrow \quad \downarrow \\
\hline
 & -4x^2 - 24x - 28 \\
 & -(-4x^2 - 16x) \downarrow \\
\hline
 & -8x - 28 \\
 & -(-8x - 32) \\
\hline
 & 4
\end{array}
$$

$$\frac{4x^3 + 12x^2 - 24x - 28}{x + 4} = 4x^2 - 4x - 8 + \frac{4}{x+4}$$

Definition

SYNTHETIC DIVISION is a shortcut that can be used when the divisor is a binomial in the form $x - k$, where k is a real number. In synthetic division, only the COEFFICIENTS are used in the division process.

Use synthetic Division to Divide Two Polynomials

1. Write k for the DIVISOR.
2. Write the coefficients of the dividend.
3. Bring down the LEADING COEFFICIENT.
4. Multiply the leading coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the REMAINDER and it has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, etc.

Example 2. Use synthetic division to divide $6x^3 - 18x^2 + 19$ by $x - 2$.

$$x - 2 = 0$$
$$x = 2$$

$$\begin{array}{r|rrrr} 2 & 6 & -18 & 0 & 19 \\ & \downarrow & 12 & -12 & -24 \\ \hline & 6 & -6 & -12 & \boxed{-5} \\ & x^2 & x & c & \end{array}$$

REMAINDER

$$\boxed{\frac{6x^3 - 18x^2 + 19}{x - 2} = 6x^2 - 6x - 12 + \left(\frac{-5}{x-2}\right)}$$

* IF WE MULTIPLY BOTH SIDES BY THE LCD, $x-2$, WE OBTAIN:

$$\left[\frac{6x^3 - 18x^2 + 19}{x - 2} = 6x^2 - 6x - 12 + \left(\frac{-5}{x-2}\right)\right](x-2)$$

$$6x^3 - 18x^2 + 19 = (6x^2 - 6x - 12)(x-2) - 5$$

Example 3. Use synthetic division to divide $16x^3 + 8x^2 - 32x - 20$ by $4x + 4$.

$$4x+4=0$$
$$4x=-4$$
$$x=-1$$

$ax+b$

$4x+4 \Rightarrow a=4$

DIVIDE BY 4

$$\begin{array}{r|rrrr} -1 & 16 & 8 & -32 & 20 \\ & \downarrow & -16 & 8 & 24 \\ \hline & 16 & -8 & -24 & \boxed{44} \end{array}$$

REMAINDER

x^2 x c

$$\frac{16x^3+8x^2-32x-20}{4x+4} = \frac{16x^2-8x-24}{4} + \frac{44}{4x+4}$$

$$\boxed{\frac{16x^3+8x^2-32x-20}{4x+4} = 4x^2-2x-6 + \frac{44}{4x+4}}$$

$$\Rightarrow 16x^3+8x^2-32x-20 = (4x^2-2x-6)(4x+4)+44$$

10.2 Possible Rational Roots

Rational Root Theorem

The possible rational roots of the polynomial

$$a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$$

are of the form $\pm\frac{p}{q}$, where p is a divisor of a_0 and q is a divisor of a_n.

CONSTANT TERM LEADING COEFFICIENT

How to Use the Rational Root Theorem to Find the Zeros of a Polynomial $f(x)$

1. Determine all divisors of the constant term a_0 and all divisors of the leading coefficient a_n.
2. Use 1 to determine all possible values of $\pm\frac{p}{q}$, where p is a divisor of a_0 and q is a divisor of a_n.
3. Determine which possible zeros are actually zeros of $f(x)$ by evaluating each case of $f\left(\pm\frac{p}{q}\right)$.

Example 4. Find the possible rational roots of the following polynomial:

$$f(x) = 6x^3 - 17x^2 + 6x + 8$$

- $a_0 = 8$ DIVISORS OF 8: 1, 2, 4, 8
- $a_n = 6$ DIVISORS OF 6: 1, 2, 3, 6

$\Rightarrow$ POSSIBLE RATIONAL ROOTS:

$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}$

$\pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{2}{3}, \pm\frac{2}{6}$

$\pm\frac{4}{1}, \pm\frac{4}{2}, \pm\frac{4}{3}, \pm\frac{4}{6}$

$\pm\frac{8}{1}, \pm\frac{8}{2}, \pm\frac{8}{3}, \pm\frac{8}{6}$

Example 5. Find the possible rational roots of the following polynomial and then find the actual roots by factoring or using the Quadratic Formula:

$$f(x) = x^2 - 15$$

· a_0 : -15 DIVISORS OF 15 : 1, 3, 5, 15

· a_n : 1 DIVISORS OF 1 : 1

$\Rightarrow$ POSSIBLE RATIONAL ZEROS : $\pm 1, \pm 3, \pm 5, \pm 15$

QUADRATIC FORMULA : $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x^2 + 0x - 15$

$$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(-15)}}{2(1)}$$

$$\boxed{x = \frac{\pm\sqrt{60}}{2}}$$

OUR ACTUAL ROOTS ARE $\pm \dfrac{\sqrt{60}}{2}$

★ OUR RATIONAL ROOT THEOREM ONLY HELPS US FIND POSSIBLE RATIONAL ROOTS, NOT IRRATIONAL ROOTS

10.3 Completely Factor Polynomials

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$ then the value of f(K) is the remainder.

The Factor Theorem

k is a zero of $f(x)$ if and only if (X-K) is a factor of $f(x)$.

How to Find the Zeros of a Polynomial $f(x)$ Using Synthetic Division

1. Use the RATIONAL ROOT THEOREM to find all of the possible rational roots (zeros) of $f(x)$.
2. Use synthetic division to evaluate a given possible zero. If the remainder is 0, the candidate is a zero. If the remainder is not 0, discard the candidate.
3. Repeat step 2 using the quotient found with synthetic division. Continue (if possible) until the quotient is a quadratic.
4. Find the zeros of the quadratic.

Example 6. Factor the polynomial below and list all of the actual zeros for the polynomial:

$$f(x) = x^3 - 6x^2 - 15x + 100$$

- $a_0 = 100$ DIVISORS OF 100: 1, 2, 4, 5, 10, 20, 25, 50, 100
- $a_n = 1$ DIVISORS OF 1: 1

$\Rightarrow$ POSSIBLE RATIONAL ROOTS: $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm 25, \pm 50, \pm 100$

TRY 1:

$$\begin{array}{r|rrrr} 1 & 1 & -6 & -15 & 100 \\ & \downarrow & 1 & -5 & -20 \\ \hline & 1 & -5 & -20 & \cancel{80} \end{array}$$

TRY −1:

$$\begin{array}{r|rrrr} -1 & 1 & -6 & -15 & 100 \\ & \downarrow & 1 & 5 & -10 \\ \hline & -1 & -5 & 10 & \cancel{90} \end{array}$$

TRY 2:

$$\begin{array}{r|rrrr} 2 & 1 & -6 & -15 & 100 \\ & \downarrow & 2 & -8 & -46 \\ \hline & 1 & -4 & -23 & \cancel{54} \end{array}$$

TRY −4:

$$\begin{array}{r|rrrr} -4 & 1 & -6 & -15 & 100 \\ & \downarrow & -4 & 40 & -100 \\ \hline & 1 & -10 & 25 & 0 \\ & x^2 & x & c & \end{array}$$

YES!

$$\Rightarrow x^3 - 6x^2 - 15x + 100 = (x+4)(x^2 - 10x + 25)$$
$$= (x+4)(x-5)(x-5)$$

$$x^2 - 10x + 25 = (x-5)(x-5)$$

Example 7. Factor the polynomial below and list all of the actual zeros for the polynomial:

$$f(x) = x^4 + 12x^3 + 37x^2 - 30x - 200$$

- $a_0 = -200$ DIVISORS OF 200: 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200
- $a_n = 1$ DIVISORS OF 1:

$\Rightarrow$ POSSIBLE RATIONAL ZEROS:

$\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 20, \pm 25, \pm 40, \pm 50, \pm 100, \pm 200$

TRY 2:

$$\begin{array}{r|rrrrr} 2 & 1 & 12 & 37 & -30 & -200 \\ & \downarrow & 2 & 28 & 130 & 200 \\ \hline & 1 & 14 & 65 & 100 & \boxed{0} \\ & x^3 & x^2 & x & c & \text{YES!} \end{array}$$

$$\Rightarrow x^4 + 12x^3 + 37x^2 - 30x - 200 = (x-2)\underbrace{(x^3 + 14x^2 + 65x + 100)}_{\text{FACTOR!}}$$

- $a_0 = 100$ DIVISORS OF 100: 1, 2, 4, 5, 10, 20, 25, 50, 100
- $a_n = 1$ DIVISORS OF 1: 1

$\Rightarrow$ POSSIBLE RATIONAL ROOTS: $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm 25, \pm 50, \pm 100$

TRY -4:

$$\begin{array}{r|rrrr} -4 & 1 & 14 & 65 & 100 \\ & \downarrow & -4 & -40 & -100 \\ \hline & 1 & 10 & 25 & \boxed{0} \\ & x^2 & x & c & \text{YES!} \end{array}$$

$$\Rightarrow x^3 + 14x^2 + 65x + 100 = (x+4)(x^2 + 10x + 25)$$
$$= (x+4)(x+5)^2$$

$$\Rightarrow x^4 + 12x^3 + 37x^2 - 30x - 200 = (x-2)(x^3 + 14x^2 + 65x + 100)$$
$$= (x-2)(x+4)(x+5)^2$$

Example 8. Factor the polynomial below and list all of the actual zeros for the polynomial:

$$f(x) = 3x^3 + 7x^2 - 11x - 15$$

- $a_0 = -15$ DIVISORS OF 15: 1, 3, 5, 15
- $a_n = 3$ DIVISORS OF 3: 1, 3

$\Rightarrow$ POSSIBLE RATIONAL ZEROS: $\pm 1, \pm \frac{1}{3}, \pm 3, \pm 5, \pm \frac{5}{3}, \pm 15$

TRY -1:

$$\begin{array}{r|rrrr} -1 & 3 & 7 & -11 & -15 \\ & \downarrow & -3 & -4 & 15 \\ \hline & 3 & 4 & -15 & \boxed{0} \end{array}$$

YES!

$3x^3 + 7x^2 - 11x - 15 = (x+1)(3x^2 + 4x - 15)$

FACTOR $3x^2 + 4x - 15$: FACTORS OF $(-15)(3) = -45$ THAT ADD TO 4: 9, −5 WORK

$$\begin{aligned} 3x^2 + 4x - 15 &= 3x^2 + 9x - 5x - 15 \\ &= 3x(x+3) - 5(x+3) \\ &= (x+3)(3x-5) \end{aligned}$$

$$\Rightarrow \boxed{3x^3 + 7x^2 - 11x - 15 = (x+1)(x+3)(3x-5)}$$

Example 9. Factor the polynomial below and list all of the actual zeros for the polynomial:

$$f(x) = 18x^3 - 9x^2 - 38x + 24$$

- $a_0 = 24$ DIVISORS OF 24: 1, 2, 3, 4, 6, 8, 12, 24
- $a_n = 18$ DIVISORS OF 18: 1, 2, 3, 6, 9, 18

$\Rightarrow$ POSSIBLE RATIONAL ZEROS:

$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{1}{9}, \pm \frac{1}{18}, \pm 2, \pm \frac{2}{3}, \pm \frac{2}{9}, \ldots$ ETC ...

TRY $\frac{2}{3}$:

$$\begin{array}{c|cccc} \frac{2}{3} & 18 & -9 & -38 & 24 \\ & \downarrow & 12 & 2 & -24 \\ \hline & 18 & 3 & -36 & \boxed{0} \\ & x^2 & x & c & \end{array}$$

YES!

$x - \frac{2}{3}$ IS A FACTOR!

$\Rightarrow 3x - 2$ IS A FACTOR

$a = 3$ SO WE NEED TO DIVIDE BY 3 STILL!

$$\Rightarrow \frac{18x^3 - 9x^2 - 38x + 24}{(3x-2)} = \frac{18x^2 + 3x - 36}{3}$$

$$\frac{18x^3 - 9x^2 - 38x + 24}{(3x-2)} = 6x^2 + x - 12$$

$$18x^3 - 9x^2 - 38x + 24 = (6x^2 + x - 12)(3x-2)$$

$$= (3x-4)(2x+3)(3x-2)$$

$6x^2 + x - 12$

$= (3x-4)(2x+3)$