Module 2 Lecture Notes

MAC1105

Fall 2019

2 Linear Functions

2.1 Construct a Linear Function From Points

Definition

An EQUATION in x is a statement that two algebraic expressions are equal.

* $3x+2$ IS NOT AN EQUATION
* $y=x+5$ IS AN EQUATION

Definition

A linear equation is the equation of a straight line written in one variable. A linear equation (in one variable) can be written in the form $ax+b=0$ where $a \neq 0$.

* $5x+2=0$ IS A LINEAR EQUATION
* $2x^2+1=0$ IS NOT A LINEAR EQUATION

Note 1. The only power of the variable in a linear equation is 1. For example, $x^2 - 4 = 0$ is not a linear equation because the power of the variable x is 2.

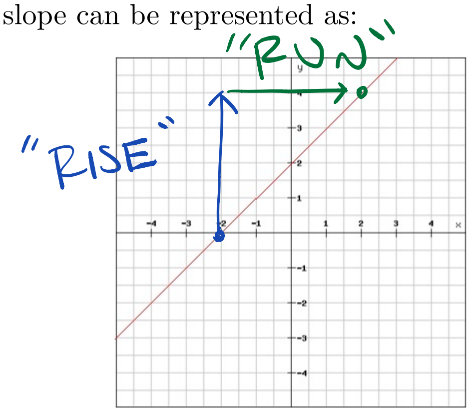
Definition

The SLOPE of a line is used to find the steepness and direction of a line. The vertical change between two points is called the RISE, and the horizontal change between two points is called the RUN. So, slope $= \dfrac{\text{rise}}{\text{run}}$. Given two pints on a line, (x_1, y_1) and (x_2, y_2), the slope of the line between the two points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

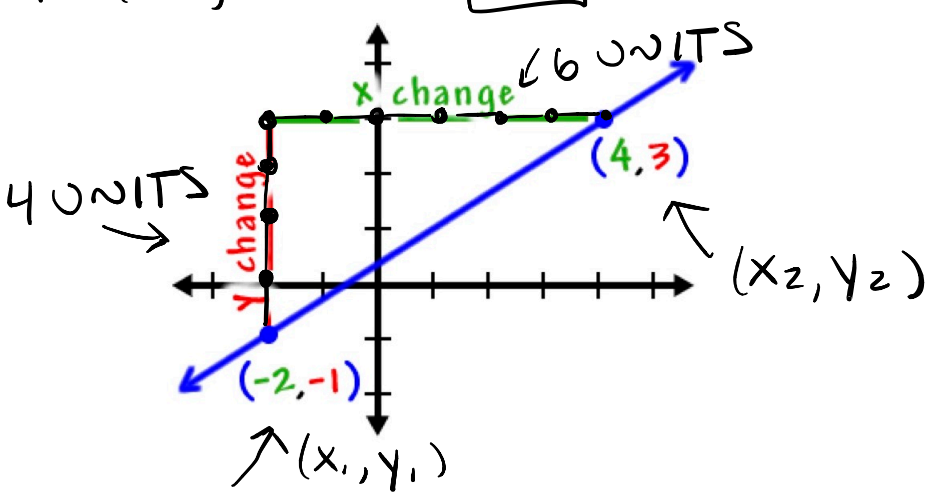
where m denotes the slope.

Note 2. On a graph, the slope can be represented as:



Example 1. What is the slope of the line below?

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-1)}{4-(-2)} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

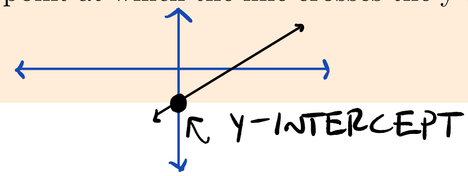


Note 3. It does not matter which point you choose as (x_1, y_1) and which you choose as (x_2, y_2).

But, make sure you stay consistent with the order of the y terms and the order of the x terms in the numerator and denominator.

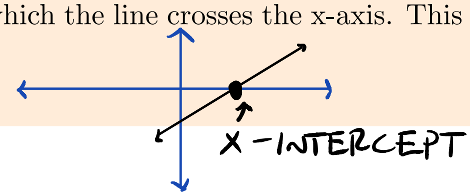
Definition

The Y - INTERCEPT of a line is the point at which the line crosses the y-axis. This is the point where $x = 0$.

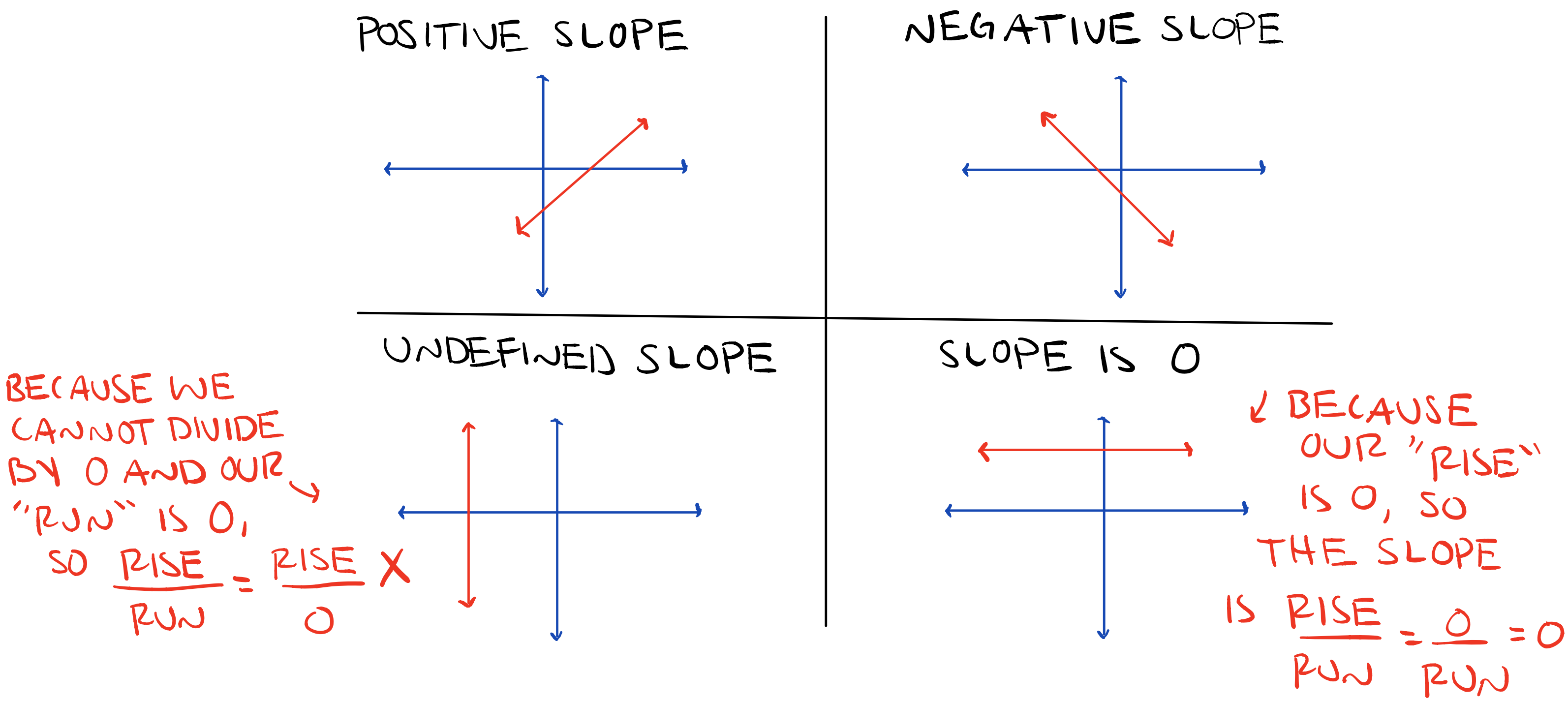


Definition

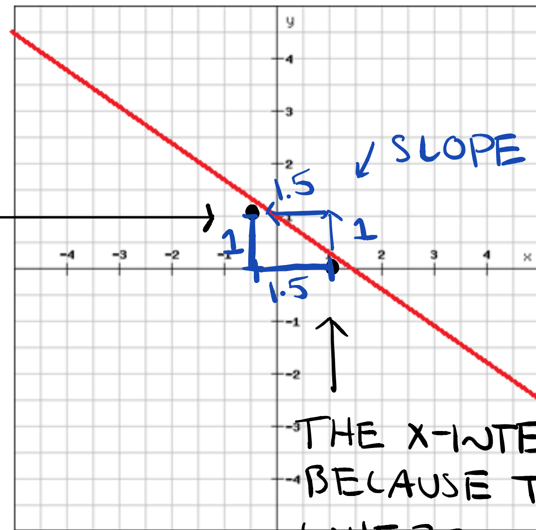
The X - INTERCEPT of a line is the point at which the line crosses the x-axis. This is the point where $y = 0$.



Note 4. The slope of a line can be positive, negative, undefined, or 0:



Example 2. Identify the slope, y-intercept, and x-intercept of the line below.



Y-INTERCEPT IS (0,1) BECAUSE THIS THE POINT WHERE THE LINE CROSSES THE Y-AXIS.

SLOPE IS $\frac{\text{RISE}}{\text{RUN}} = -\frac{1}{1.5}$

NEGATIVE!

THE X-INTERCEPT IS (1.5,0) BECAUSE THIS IS THE POINT WHERE THE LINE CROSSES THE X-AXIS

Slope-Intercept Form

The slope-intercept form of an equation is $y = mx + b$, where m is the SLOPE and b is the Y - INTERCEPT

Note 5. If we are given an equation that is not in slope-intercept form, we can still always find the y-intercept by plugging in $x = 0$ and solving for y.

Example 3. Find the equation of the line containing the two points:

$$\underset{(x_1,y_1)}{(2,-1)}, \quad \underset{(x_2,y_2)}{(3,4)}$$

1. FIND THE SLOPE: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-1)}{3-2} = \frac{5}{1} = 5$

2. WRITE IN THE FORM $y = mx + b$: $\boxed{m=5}$, SO $y = 5x + b$

3. SUBSTITUE IN A POINT TO SOLVE FOR b: I WILL USE (2,−1)

$y = 5x + b$

$-1 = 5(2) + b \Rightarrow -1 = 10 + b$

$-10 = -10$

$\boxed{-11 = b}$

$\Rightarrow y = 5x + (-11)$

$\boxed{y = 5x - 11}$

Example 4. Find the equation of the line containing the two points:

$$\underset{(x_2, y_2)}{(-2, 3)}, \quad \underset{(x_1, y_1)}{(6, -5)}$$

1. FIND THE SLOPE: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3 - (-5)}{-2 - 6} = \dfrac{8}{-8} = -1$

2. WRITE AS $y = mx + b$: $\boxed{m = -1}$, SO $y = (-1)x + b$ OR $y = -x + b$

3. SUBSTITUTE A POINT TO SOLVE FOR b: I WILL USE $(-2, 3)$

$y = -x + b$

$3 = -(-2) + b$

$3 = 2 + b$

$-2 \quad = -2$

$\boxed{1 = b}$

$y = -x + b \Rightarrow \boxed{y = -x + 1}$

Point-Slope Form

Given the slope and one point on a line, we can find the equation of the line using point-slope form:

$$y - y_1 = m(x - x_1)$$

where m is the SLOPE and (x_1, y_1) is the point that we are given.

Definitions

- Two lines are PARALLEL if they have the same slope but have different y-intercepts.
- Two lines are PERPENDICULAR if the slope of one line is the negative reciprocal of the other.

Note 6. To find the negative reciprocal of a number, put the number in fraction form and switch the numerator and denominator, then negate the number. For example,

1. $-\frac{1}{5} \longleftrightarrow -\left(-\frac{5}{1}\right) = 5$

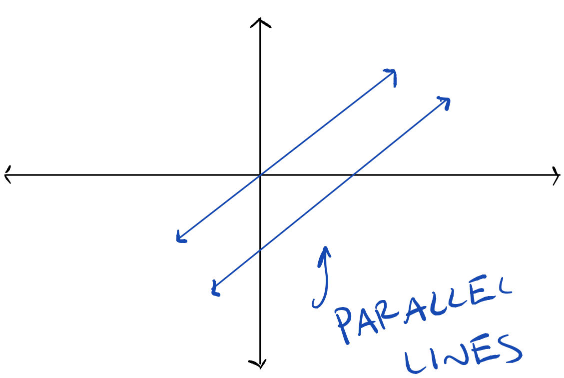
2. $\frac{4}{3} \longleftrightarrow -\left(\frac{3}{4}\right) = -\frac{3}{4}$

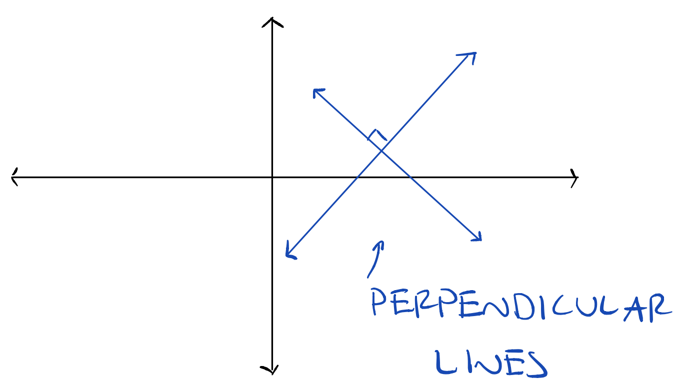
How to Find the Equation of Parallel and Perpendicular Lines:

1. Determine the slope of the given line. I suggest writing the equation in slope-intercept form first.

2. If you are finding a parallel line, the slope is the same as the original line. If you are finding a perpendicular line, the slope is the NEGATIVE RECIPROCAL of the original line.

3. Use the point you are given and the slope you found in step 2 to determine the equation of the line.





Example 5. Write the equation of a line that is parallel to $y = 6x + 1$ and passes through the point $(-7, 1)$.

$m=6$

* PARALLEL ⇒ SAME SLOPE!

SINCE THE SLOPE IS $m=6$, THE PARALLEL LINE HAS THE FORM $y=6x+b$.

SUBSTITUTE TO SOLVE FOR b

$$1 = 6(-7)+b$$
$$1 = -42 + b$$
$$+42 = +42$$
$$\boxed{43 = b}$$

$$\boxed{y = 6x + 43}$$

Example 6. Write the equation of a line that is perpendicular to $x + 3y = 6$ and passes through the point $(1, 5)$.

1. WRITE IN SLOPE-INTERCEPT FORM:

$$x + 3y = 6$$
$$-x \quad = -x$$
$$\frac{3y}{3} = \frac{6-x}{3}$$
$$y = \frac{6-x}{3}$$

$$y = \frac{6}{3} - \frac{x}{3}$$
$$y = 2 - \frac{1}{3}x$$
$$\underbrace{y = -\frac{1}{3}x + 2}$$

SLOPE-INTERCEPT FORM!
SLOPE IS $m = -\frac{1}{3}$

2. FIND NEW SLOPE: PERPENDICULAR ⇒ NEGATIVE RECIPROCAL

OLD $m = -\frac{1}{3}$ ⇒ NEW $\boxed{m = 3}$

3. USE POINT-SLOPE FORM WITH $m=3$, $(x_1, y_1) = (1, 5)$

$$y - y_1 = m(x - x_1)$$
$$y - 5 = 3(x - 1)$$
$$y - 5 = 3x - 3$$
$$+5 = \quad +5$$
$$\boxed{y = 3x + 2}$$

2.2 Converting Between Linear Forms

The Standard Form of a Line

The standard form of a line is $Ax + By = C$ where A, B, and C are integers.

Note 7. When the equation of a line is written in standard form, the x and y terms are on one side of the equal sign, and the constant term is on the other side of the equal sign.

Example 7. Convert the linear function below from standard form to slope-intercept form:

*SLOPE-INTERCEPT FORM IS $y = mx + b$

$$7x + 5y = 2.$$

$$-7x \quad = -7x$$

$$\frac{5y}{5} = \frac{-7x+2}{5}$$

$$y = \frac{-7x+2}{5} \Rightarrow \boxed{y = -\frac{7}{5}x + \frac{2}{5}}$$

Example 8. Convert the linear function below from slope-intercept form to standard form:

$$y = -\frac{5}{4}x - \frac{5}{2}.$$

WANT: $Ax + By = C$ (NO FRACTIONS!)

$$y = -\frac{5}{4}x - \frac{5}{2}$$

$$+\frac{5}{4}x = +\frac{5}{4}x$$

$$\left(\frac{5}{4}x + y = -\frac{5}{2}\right)4$$

MULTIPLY BY LCD TO GET RID OF FRACTIONS

$$4\left(\frac{5}{4}x + y\right) = 4\left(-\frac{5}{2}\right)$$

$$5x + 4y = 2(-5)$$

$$\boxed{5x + 4y = -10}$$

2.3 Convert Between a Linear Equation and its Graph

How to Determine the Equation of a Line Given its Graph

To find the equation of a line in slope-intercept form given its graph,

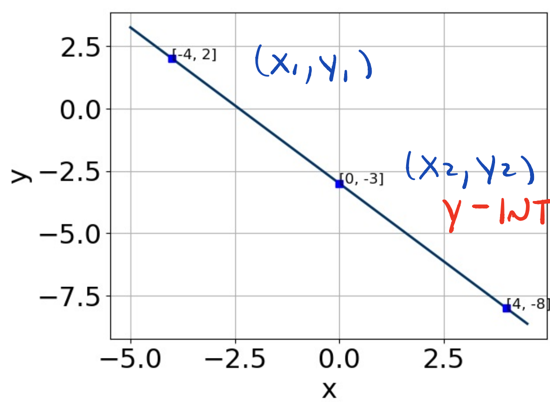
1. Find the slope of the line using the formula for the slope between two points.
2. Use the slope and any point on the graph to find the y-intercept, or find the y-intercept on the graph.
3. Write the equation in the form $y = mx + b$.

Example 9. Write the equation of the line below in slope-intercept form and in standard form.

1. FIND SLOPE:

$m = \dfrac{-3-2}{0-(-4)} = \dfrac{-5}{4}$

$\boxed{m = -\dfrac{5}{4}}$



(x_1, y_1)

(x_2, y_2)

Y-INTERCEPT IS WHERE $x = 0$, SO Y-INTERCEPT IS $\boxed{b = -3.}$

$\boxed{y = -\dfrac{5}{4}x - 3}$

2. USE GRAPH TO FIND Y-INTERCEPT, OR SUBSTITUTE $(4, -8)$ IN TO $y = -\frac{5}{4}x + b$ TO SOLVE FOR b:

$-8 = -\frac{5}{4}(4) + b$

$-8 = -5 + b$

$+5 = +5$

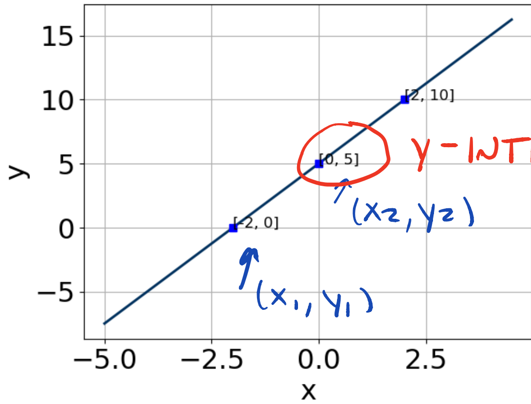
$\boxed{-3 = b}$ $\boxed{y = -\dfrac{5}{4}x - 3}$

Example 10. Write the equation of the line below in slope-intercept form and in standard form.

1. FIND SLOPE:

$m = \frac{5-0}{0-(-2)} = \frac{5}{2}$

$\boxed{m = \frac{5}{2}}$



Y-INTERCEPT IS WHERE $x=0$, SO Y-INTERCEPT IS $\boxed{b=5}$

$y = \frac{5}{2}x + 5$

2. USE GRAPH TO FIND Y-INTERCEPT, OR SUBSTITUTE $(2,10)$ IN TO $y = \frac{5}{2}x + b$ TO SOLVE FOR b:

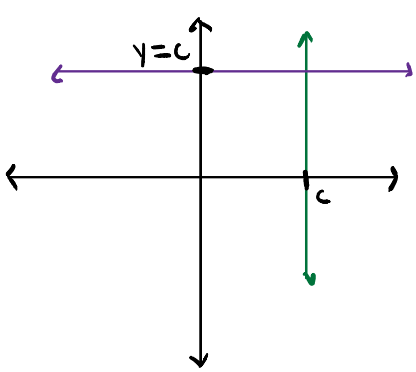
$10 = \frac{5}{2}(2) + b$

$10 = 5 + b$

$-5 = -5$

$\boxed{5 = b}$ $\boxed{y = \frac{5}{2}x + 5}$

Note 8. The equation of a vertical line is given by: $x = c$. The equation of a horizontal line is given by: $y = c$.



2.4 Solve Linear Equations

Note 9. To solve an equation is to find all values of x for which the equation is true. Such values of x are SOLUTIONS (or roots, zeros) of the equation.

How to Solve Linear Equations in One Variable The steps below do not need to be followed in any particular order (as long as you remember PEMDAS)

1. Add, subtract, multiply, or divide an equation by a number or expression. BUT, you MUST do the same thing to both sides of the equal sign. (Math is fair)
2. Apply the distributive property (you may not always need to do this): $a(b+c)=ab+ac$
3. Isolate the variable on one side of the equation.
4. Solve for the variable by dividing or multiplying a constant.

Example 11. Solve the following equation:

$$-15(2x+10) = -11(-6x+14)$$

DISTRIBUTIVE PROPERTY

$$-15(2x)+(-15)(10) = (-11)(-6x)+(-11)(14)$$

$$-30x+(-150) = 66x+(-154)$$

$$-30x-150 = 66x-154$$

$$+30x \qquad = +30x$$

GET "x" ON ONE SIDE

$$-150 = 96x-154$$

$$+154 = \qquad +154$$

$$\frac{4}{96} = \frac{96x}{96}$$

$$\frac{4}{96} = x \xrightarrow{\text{SIMPLIFY}} \boxed{\frac{1}{24} = x}$$

Example 12. Solve the following equation:

$$-2(9x-4) = -15(12x+10)$$

$$-2(9x) + (-2)(-4) = (-15)(12x) + (-15)(10)$$

$$-18x + 8 = -180x - 150$$
$$+180x \quad = +180x$$

$$162x + 8 = -150$$
$$-8 = -8$$

$$\frac{162x}{162} = \frac{-158}{162}$$

$$x = -\frac{158}{162} \longrightarrow \boxed{x = \frac{-79}{81}}$$

Definition

- A RATIONAL EXPRESSION is the quotient (or ratio) of two polynomials.
- A RATIONAL EQUATION is an equation that contains at least one rational expression.

Note 10. Recall that a rational number is the ratio of two integers. For example, $\frac{3}{2}$ and $\frac{5}{6}$ are rational numbers.

Note 11. To solve a linear equation with fractions, multiply both sides by the LEAST COMMON DENOMINATOR to clear the fraction.

Note 12. If you are given a rational equation in the form of a proportion:

$$\frac{a}{b} = \frac{c}{d}$$

then you can solve the equation by cross multiplication:

$$\frac{a}{b} \times \frac{c}{d} \qquad ad = bc$$

Example 13. Solve the equation:

$$\frac{x-3}{7} = \frac{4x+12}{7}$$

USE CROSS MULTIPLICATION!

$7(x-3) = 7(4x+12)$

$7x - 21 = 28x + 84$

$-7x \qquad = -7x$

$-21 = 21x + 84$

$-84 = \qquad -84$

$\frac{-105}{21} = \frac{21x}{21}$

$\boxed{-5 = x}$

Example 14. Solve the equation:

$$\frac{-4x-5}{3} - \frac{-5x+6}{2} = \frac{3x+8}{5}$$

1. FIND THE LCD: LCD IS $3 \cdot 2 \cdot 5 = 30$
2. MULPIPLY BY THE LCD (30): $30\left(\frac{-4x-5}{3} - \frac{-5x+6}{2} = \frac{3x+8}{5}\right)$

$$\overset{10}{\cancel{30}}\left(\frac{-4x-5}{\cancel{3}}\right) - \overset{15}{\cancel{30}}\left(\frac{-5x+6}{\cancel{2}}\right) = \overset{6}{\cancel{30}}\left(\frac{3x+8}{\cancel{5}}\right)$$

$$10(-4x-5) - 15(-5x+6) = 6(3x+8)$$

$$10(-4x) + 10(-5) + (-15)(-5x) + (-15)(6) = 6(3x) + 6(8)$$

$$-40x - 50 + 75x - 90 = 18x + 48$$

COMBINE LIKE TERMS, ie $-40x + 70x = 35x$ & $-50 - 90 = -140$

$$35x - 140 = 18x + 48$$
$$-18x \qquad = -18x$$

$$17x - 140 = 48$$
$$+140 = +140$$

$$\frac{17x}{17} = \frac{188}{17}$$

$$\boxed{x = \frac{188}{17}}$$

Example 15. Solve the equation:

$$\frac{8x-7}{4} - \frac{-7x+7}{6} = \frac{-7x+3}{3}$$

1. FIND THE LEAST COMMON DENOMINATOR: LCD IS 12

2. MULTIPLY BY LCD(12):

$$12\left(\frac{8x-7}{4} - \frac{-7x+7}{6} = -\frac{7x+3}{3}\right)$$

$$\overset{3}{\cancel{12}}\left(\frac{8x-7}{\cancel{4}}\right) - \overset{2}{\cancel{12}}\left(\frac{-7x+7}{\cancel{6}}\right) = \overset{4}{\cancel{12}}\left(-\frac{7x+3}{\cancel{3}}\right)$$

$$3(8x-7) - 2(-7x+7) = 4(-7x+3)$$

$$24x - 21 + 14x - 14 = -28x + 12$$

COMBINE LIKE TERMS, ie $24x + 14x = 38x$ & $-21 - 14 = -35$

$$38x - 35 = -28x + 12$$
$$+28x \quad = +28x$$

$$66x - 35 = 12$$
$$+35 = +35$$

$$\frac{66x}{66} = \frac{47}{66}$$

$$\boxed{x = \frac{47}{66}}$$