Module 3 Lecture Notes

MAC1105

Fall 2019

3 Linear Inequalities

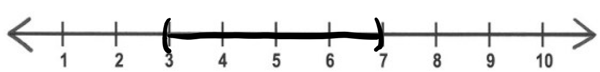
3.1 Set Notation

Definition

A set is a collection of MATHEMATICAL OBJECTS. An ELEMENT is an object that is in a specified set. An interval is a collection of REAL NUMBERS.

Example 1. The set of whole numbers is $\{0, 1, 2, \ldots\}$.

Example 2. (3,7) is the set of real numbers between 3 and 7 but NOT INCLUDING 3 and 7. On a number line, this looks like:



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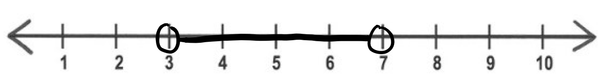
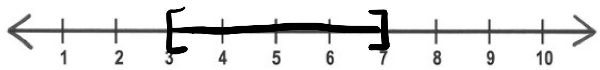


If we want to include the numbers 3 and 7 we use closed brackets, [3,7]. On a number line, this looks like:



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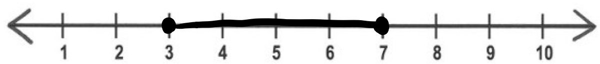
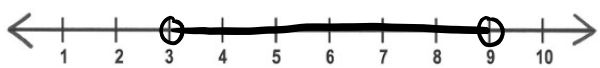


Note 1. There are a couple of different ways to describe elements in a specific set. To describe solutions that exist in an interval, we can use **interval notation** $x \in (a,b)$. We read this as "x is an element of (a,b)", or x is some number between a and b, but is not a or b. On a number line, this looks like:

$x \in (3,9)$ OR "X IS SOME NUMBER BETWEEN 3 AND 9, BUT IS NOT 3 OR 9"



To write this in **inequality notation**, we write $a < x < b$. We can also describe the elements of a set using **set builder notation**. An example of this is shown below.

Example 3. Imagine we tried to create the set of all presidents. That's a long and tedious list! But if we want to implicitly write this set, we can say $\{x : x \text{ is a president}\}$. This means THE SET OF ALL X SUCH THAT X IS A PRESIDENT. If we wanted to describe the set of all students in this course (in set builder notation), we would write

$\{x : x \text{ IS A STUDENT IN THIS COURSE}\}$.

Definition

If a and b are real numbers, then an INTERVAL, denoted (a,b) between a and b is the collection of all real numbers x such that $a < x$ and $x < b$.

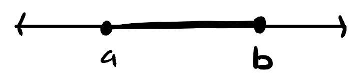
Note 2. Recall the inequality symbols and their meaning:

Symbol	Meaning
$\leq$	"LESS THAN OR EQUAL TO" Ex: $3 \leq 25$, $3 \leq 3$
$\geq$	"GREATER THAN OR EQUAL TO" Ex: $25 \geq 3$, $3 \geq 3$
$<$	"LESS THAN" Ex: $2 < 5$
$>$	"GREATER THAN" Ex: $5 > 2$

Example 4. Write the following set in interval notation and inequality notation: all real numbers between a and b and including a and b:

$\{x : x \in [a, b]\}$

$a \leq x \leq b$



Example 5. Write the following set in interval notation and inequality notation: all real numbers greater than a but not including a:

$\{x : x \in (a, \infty)\}$

$x > a$

Example 6. Write the following set in interval notation and inequality notation: all real numbers less than a or greater than b:

$x < a$ OR $x > b$

$\{x : x \in (a, \infty) \cup (b, \infty)\}$ $\cup \Rightarrow$ "OR"

Example 7. Solve the linear equation below and choose the interval that contains the solution:

$$x + 3 = 5.5$$

$$-3 = -3$$

$$x = 2.5$$

(a) $x = a$, where $a \in [-2, -1]$

(b) $x = a$, where $a \in [-1, 0]$

$x = a$, WHERE $a = 2.5$

(c) $x = a$, where $a \in [0, 1]$

(d) $x = a$, where $a \in [1, 2]$

(e) $x = a$, where $a \in [2, 3]$

3.2 Solve Linear Inequalities

Properties of Inequalities

Addition Property: If $a < b$, then $a + c < b + c$

Multiplication Property: If $a < b$ and $c > 0$, then $ac < bc$. If $a < b$ and $c < 0$, then $ac > bc$.

Note 3. The above properties also apply to $a \leq b, a > b$, and $a \geq b$.

Example 8. Illustrate the addition property by solving the following inequality:

$$x - 15 < 4$$

$$x - 15 + 15 < 4 + 15$$

$$\boxed{x < 19}$$

Example 9. Illustrate the multiplication property by solving the following inequality:

$$3x < 6$$

$$3x \cdot \frac{1}{3} < 6 \cdot \frac{1}{3}$$

$$\boxed{x < 2}$$

Note 4. We can solve linear inequalities similar to solving linear equations by combining like terms, performing operations, and isolating the variable on one side of the inequality.

Note 5. When solving an inequality, if you multiply or divide by a negative number you must FLIP THE INEQUALITY.

1. $-x < 2 \Rightarrow x > -2$

2. $-3x < 9 \Rightarrow x > -3$

Example 10. Solve the linear inequality and write your answer in inequality notation and interval notation. It may help to graph the solution on a number line:

$$-10x - 10 \leq 9x + 8$$

$$-9x \qquad = -9x$$

$$-19x - 10 \leq 8$$

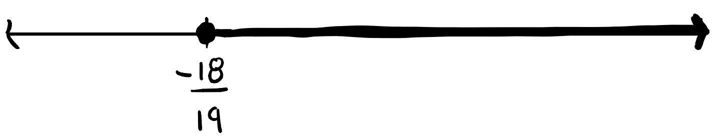
$$+10 = +10$$

$$-19x \leq 18$$

$$\frac{-19x}{-19} \quad \frac{18}{-19}$$

MUST FLIP INEQUALITY!

$$x \geq -\frac{18}{19}$$



INEQUALITY NOTATION: $x \geq \frac{-18}{19}$

INTERVAL NOTATION: $\left(-\frac{18}{19}, \infty\right)$

Example 11. Solve the linear inequality and write your answer in inequality notation and interval notation. It may help to graph the solution on a number line: LCD: 24

$$\left(x - \frac{9}{8} < \frac{7}{2}x - \frac{5}{3}\right)24$$

$$24x - \overset{3}{\cancel{24}}\left(\frac{9}{\cancel{8}}\right) < \overset{12}{\cancel{24}}\left(\frac{7}{\cancel{2}}x\right) - \overset{8}{\cancel{24}}\left(\frac{5}{\cancel{3}}\right)$$

$$24x - 27 < 84x - 40$$

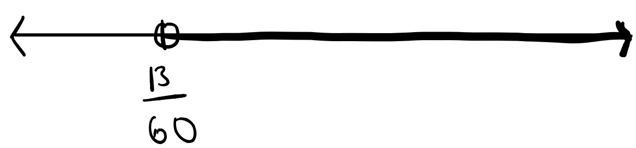
$$-24x \qquad = -24x$$

$$-27 < 60x - 40$$

$$+40 \quad = \quad +40$$

$$\frac{13}{60} < \frac{60x}{60}$$

$$\boxed{\frac{13}{60} < x}$$



$$\boxed{\left(\frac{13}{60}, \infty\right)}$$

Example 12. Solve the linear inequality and write your answer in inequality notation and interval notation. It may help to graph the solution on a number line:

$7 \cdot 4 \cdot 3 = 84$

$\Rightarrow$ COMMON DENOMINATOR : 84

$$-x - \frac{5}{4} \geq \frac{5}{3}x + \frac{8}{7}$$

$$\left(-x - \frac{5}{4} \geq \frac{5}{3}x + \frac{8}{7}\right)84$$

$$84(-x) - \overset{21}{\cancel{84}}\left(\frac{5}{\cancel{4}}\right) \geq \overset{28}{\cancel{84}}\left(\frac{5}{\cancel{3}}x\right) + \overset{12}{\cancel{84}}\left(\frac{8}{\cancel{7}}\right)$$

$$-84x - 105 \geq 140x + 96$$

$$+84x \qquad = \qquad +84x$$

$$-105 \geq 224x + 96$$

$$-96 \qquad = \qquad -96$$

$$\frac{-201}{224} \geq \frac{224x}{224}$$

$$\boxed{-\frac{201}{224} \geq x}$$

$$\boxed{\left(-\infty, -\frac{201}{224}\right]}$$

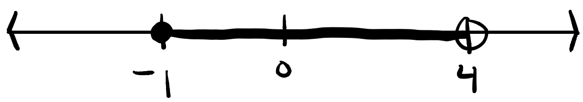
3.3 Solve Compound Linear Inequalities

Definition

A compound inequality includes TWO INEQUALITIES in one statement.

Example 13. The inequality $-1 \leq x < 4$ is a compound inequality. It means that $x \geq -1$ and $x < 4$. On a number line, this looks like

$x \in [-1, 4)$



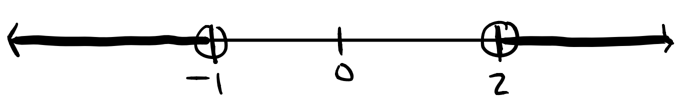
Definition

Let A and B be sets.

- The UNION of A and B, denoted $A \cup B$ is the set of all objects x such that either $x \in A$ or $x \in B$.
- The INTERSECTION of A and B, denoted $A \cap B$ is the set of all objects x such that $x \in A$ and $x \in B$.

Example 14. Represent the following expression on the number line:

$$x < -1 \quad \text{or} \quad x > 2$$



$x \in (-\infty, -1) \cup (2, \infty)$

Note 6. To solve compound inequalities, first split the inequality into two parts. Then, solve each inequality separately. Finally, put the two inequalities back together after solving.

Example 15. Solve the following inequality and write your answer in inequality notation and interval notation:

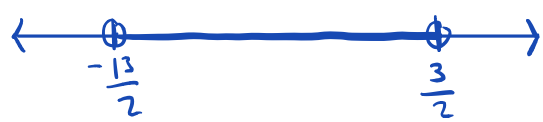
$$5x - 8 < 6x - \frac{3}{2} < 3x + 3$$

$$5x - 8 < 6x - \frac{3}{2}$$
$$\underline{-5x \qquad -5x}$$
$$-8 < x - \frac{3}{2}$$
$$\underline{+\frac{3}{2} \qquad +\frac{3}{2}}$$
$$-8 + \frac{3}{2} < x$$
$$-\frac{16}{2} + \frac{3}{2} < x$$
$$-\frac{13}{2} < x$$

AND

$$6x - \frac{3}{2} < 3x + 3$$
$$\underline{-3x \qquad -3x}$$
$$3x - \frac{3}{2} < 3$$
$$\underline{+\frac{3}{2} \qquad +\frac{3}{2}}$$
$$3x < 3 + \frac{3}{2}$$
$$3x < \frac{6}{2} + \frac{3}{2}$$
$$\left(\frac{1}{3}\right)(3x) < \left(\frac{9}{2}\right)\left(\frac{1}{3}\right)$$
$$x < \frac{9}{6}$$
$$x < \frac{3}{2}$$

$$-\frac{13}{2} < x \text{ AND } x < \frac{3}{2}$$



$$\boxed{-\frac{13}{2} < x < \frac{3}{2}}$$

$$\boxed{x \in \left(-\frac{13}{2}, \frac{3}{2}\right)}$$

Example 16. Solve the following inequality and write your answer in inequality notation and interval notation:

$$-x - \frac{1}{2} < -\frac{5}{4}x + 3 \quad \text{or} \quad \frac{7}{2}x + 1 > \frac{8}{5}x - \frac{5}{6}$$

LCD: 4

$\left(-x - \frac{1}{2} < -\frac{5}{4}x + 3\right)4$

$4\left(-x - \frac{1}{2}\right) < 4\left(-\frac{5}{4}x + 3\right)$

$-4x - 2 < -5x + 12$

$+5x \qquad +5x$

$x - 2 < 12$

$+2 \qquad +2$

$x < 14$

OR

LCD: 30

$\left(\frac{7}{2}x + 1 > \frac{8}{5}x - \frac{5}{6}\right)30$

$30\left(\frac{7}{2}x + 1\right) > 30\left(\frac{8}{5}x - \frac{5}{6}\right)$

$105x + 30 > 48x - 25$

$-48x \qquad -48x$

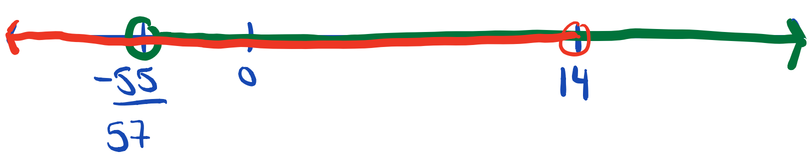
$57x + 30 > -25$

$-30 \qquad -30$

$\frac{57x}{57} > \frac{-55}{57}$

$x > -\frac{55}{57}$

$x < 14$ OR $x > -\frac{55}{57}$



Example 17. Solve the following inequality and write your answer in inequality notation and interval notation:

LCD: 8

$$\left(-8x - 4 \leq \frac{17}{4}x + \frac{3}{8} \leq 4x + 3\right) 8$$

$$8(-8x-4) \leq 8\left(\frac{17}{4}x + \frac{3}{8}\right) \leq 8(4x+3)$$

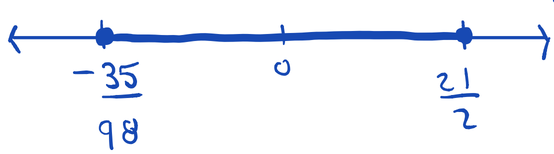
$$-64x - 32 \leq 34x + 3 \leq 32x + 24$$

$$\begin{aligned} -64x - 32 &\leq 34x + 3 \\ +64x \qquad &\qquad +64x \\ \hline -32 &\leq 98x + 3 \\ -3 \qquad &\qquad -3 \\ \hline \frac{-35}{98} &\leq \frac{98x}{98} \\ -\frac{35}{98} &\leq x \end{aligned}$$

AND

$$\begin{aligned} 34x + 3 &\leq 32x + 24 \\ -32x \qquad &\qquad -32x \\ \hline 2x + 3 &\leq 24 \\ -3 \qquad &\qquad -3 \\ \hline \frac{2x}{2} &\leq \frac{21}{2} \\ x &\leq \frac{21}{2} \end{aligned}$$

$$-\frac{35}{98} \leq x \text{ AND } x \leq \frac{21}{2}$$



$$\boxed{-\frac{35}{98} \leq x \leq \frac{21}{2}}$$

$$\boxed{x \in \left[-\frac{35}{98}, \frac{21}{2}\right]}$$