Module 4 Lecture Notes

MAC1105

Summer B 2019

4 Quadratic Functions

4.1 Factor Trinomials

Rules for Positive Exponents

For all positive integers m and n and all real numbers a and b:

Product Rule

$a^n a^m = a^{n+m}$

Power Rules

$(a^n)^m = a^{n \cdot m}$

$(ab)^{nm} = a^{nm} b^{nm}$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad b \neq 0$

Zero Exponent

$a^0 = 1$

Definition

An expression of the form $a_k x^k$ where $k \geq 0$ is an integer, a_k is a constant, and x is a variable, is called a MONOMIAL. The constant a_k is called the COEFFICIENT and k is the DEGREE of the monomial if $k \neq 0$.

Note 1. The sum of monomials with different degrees forms a POLYNOMIAL. The monomials in the polynomial are called the TERMS. A polynomial with exactly two terms is called a BINOMIAL and a polynomial with exactly 3 terms is called a TRINOMIAL.

Polynomial in One Variable in Standard Form:

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

where $a_0, ..., a_{n-1}, a_n$ are real numbers and $n \geq 0$ is an integer.

Ex: $5x^5 + 7x^4 + 17x^3 + 12x^2 + x + 29$

$21x^{17} + 1$

$x^2 + 12x$

Definition

A QUADRATIC FUNCTION is a polynomial function of degree 2.

Note 2. $x^2 + 5x + 2$ is a quadratic function, but $x + 2$ is not a quadratic function because THE DEGREE OF X+2 IS 1, NOT 2.

Operations on Polynomials

Adding and Subtracting Polynomials

Polynomials are added and subtracted by combining like terms.

Multiplying Polynomials

Two polynomials are multiplied by using the properties of real numbers and the rules for exponents.

Example 1. Perform the operation:

$$(2x^4 - 3x^2 + 1)(4x - 1)$$

$= (2x^4)(4x) + (2x^4)(-1) + (-3x^2)(4x) + (-3x^2)(-1) + (1)(4x) + (1)(-1)$

$= 8x^4x - 2x^4 - 12x^2x + 3x^2 + 4x - 1$

$= 8x^5 - 2x^4 - 12x^3 + 3x^2 + 4x - 1$

Note 3. When multiplying two binomials, use FOIL (OR "THE BOX METHOD")

"FIRST, OUTER, INNER, LAST"

EXAMPLE: $(x+2)(x-2)$

$= x \cdot x + (-2)(x) + (2)(x) + (2)(-2)$

$= x^2 - 2x + 2x - 4 = \boxed{x^2 - 4}$

	x	2
x	x^2	$2x$
-2	$-2x$	-4

$= x^2 + 2x + (-2x) + (-4)$

$= x^2 - 4$

Definition

The greatest common factor (GCF) of a polynomial is the LARGEST POLYNOMIAL that divides evenly into the polynomials.

Ex: THE GCF OF $16x$ AND $20x^2$ IS $4x$

THE GCF OF $2x^2 + 4x + 22$ IS 2

How to Factor out the Greatest Common Factor

1. Identify the GCF of the COEFFICIENTS
2. Identify the GCF of the VARIABLES.
3. Combine 1 and 2 to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the polynomial.
5. Write the factored polynomial as the product of the GCF and the sum of the terms we need to multiply by.

Example 2. Factor $6x^3y^3 + 45x^2y^2 + 21xy$ by factoring out the greatest common factor.

1. GCF OF THE COEFFICIENTS (6, 45, 21) IS 3
2. GCF OF THE VARIABLES (x^3y^3, x^2y^2, xy) IS xy
3. THE GCF OF THE EXPRESSION IS $3 \cdot xy = 3xy$
4. FACTOR OUT THE GCF: $3xy(2x^2y^2 + 15xy + 7)$

Factor a Trinomaial with Leading Coefficient 1

A trinomial of the form $x^2 + bx + c$ can be factored as $(x + p)(x + q)$, where $pq =$ c and $p + q =$ b.

Note 4. Not every polynomial can be factored. Some polynomials cannot be factored, in which case we say the polynomial is prime.

How to Factor a Trinomial of the Form $x^2 + bx + c$

1. Determine all possible factors of c.
2. Using the list found in 1, find two factors p and q, in which $pq =$ c and $p + q =$ b.
3. Write the factored expression as $(x+p)(x+q)$.

Note 5. The order in which you write the factored polynomial does not matter. This is because multiplication is COMMUTATIVE.

Example 3. Factor the trinomial:

$$x^2 + 24x + 140$$

1. ALL POSSIBLE FACTORS OF 140:

140	1
70	2
35	4
28	5
20	7
14	10

2. FIND TWO FACTORS P AND q, FOUND IN (1) WHERE $p+q=24$

$\Rightarrow 14+10=24$

3. $(x+14)(x+10)$

☆ CAN CHECK OUR ANSWER BY FOIL!

$(x+14)(x+10) = x^2 + 10x + 14x + 140$

$= x^2 + 24x + 140$ ✓

Factor a Trinomial by Grouping

To factor a trinomial in the form $ax^2 + bx + c$ by grouping, we find two numbers with a product of ac and a sum of b.

How to Factor a Trinomial of the Form $ax^2 + bx + c$ by Grouping

1. Determine all possible factors of ac.
2. Using the list found in 1, find two factors p and q, in which $pq =$ ac and $p + q =$ b.
3. Rewrite the original polynomial as $ax^2+px+qx+c$.
4. Pull out the GCF of ax^2+px.
5. Pull out the GCF of $qx+c$.
6. Factor out the GCF of the expression.

Example 4. Factor the polynomial:

$$35x^2 + 48x + 16$$

★ FIND THE FACTORS OF $35 \cdot 16 = 560$ WHOSE SUM IS 48:

560	1
280	2
140	4
112	5
80	7
70	8
56	10
40	14
35	16
20	28

$20 + 28 = 48$!

$\Rightarrow$ REWRITE AS: $\underbrace{35x^2 + 20x}_{\text{PULL OUT GCF}} + \underbrace{28x + 16}_{\text{PULL OUT GCF}}$

$= 5x(7x + 4) + 4(7x + 4)$

GCF OF EXPRESSION

$= \boxed{(7x+4)(5x+4)}$

Example 5. Factor the polynomial:

$$21x^2 + 40x + 16$$

★ FIND THE FACTORS OF $(21)(16) = 336$ WHOSE SUM IS 40:

336	1
168	2
112	3
84	4
56	6
48	7
42	8
28	12

$28 + 12 = 40$!

$\Rightarrow$ REWRITE AS: $\underbrace{21x^2 + 28x}_{\text{PULL OUT GCF}} + \underbrace{12x + 16}_{\text{PULL OUT GCF}}$

$= 7x(3x + 4) + 4(3x + 4)$

GCF OF EXPRESSION

$= \boxed{(3x+4)(7x+4)}$

Example 6. Factor the polynomial:

$$36x^2 + 19x - 6$$

★ FIND FACTORS OF $(36)(-6) = -216$ WHOSE SUM IS 19:

-216	1	-27	8
216	-1	27	-8
-108	2		
108	-2		
-72	3		
72	-3		
-54	4		
54	-4		
-36	6		
36	-6		

$27 - 8 = 19$

$\Rightarrow$ REWRITE AS: $\underbrace{36x^2 + 27x}_{\text{PULL OUT GCF}} \underbrace{- 8x - 6}_{\text{PULL OUT GCF}}$

$= 9x(4x + 3) - 2(4x + 3)$

GCF OF EXPRESSION

$= \boxed{(4x+3)(9x-2)}$

Factor a Perfect Square Trinomial

A perfect square trinomial can be written as the square of a binomial:

$$a^2+2ab+b^2 = (a+b)^2$$

or

$$a^2-2ab+b^2 = (a-b)^2$$

How to Factor a Perfect Square Trinomial

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is TWICE the product of ab.
3. Write the factored form as $(a+b)^2$.

Example 7. Factor the polynomial:

$$100x^2 + 60x + 9$$

★ FIRST AND LAST TERM ARE PERFECT SQUARES $100x^2 = (10x)^2$ AND $9 = 3^2$

Q: IS THE MIDDLE TERM, $6x$, TWICE THE PRODUCT OF $(10x)(3)$?

$$2(10x)(3) = (20x)(3)$$
$$= 60x \checkmark$$

YES!

⇒ PERFECT SQUARE TRINOMIAL:

$$100x^2+60x+9 = (10x)^2 + 2(10x)(3) + 3^2 = \boxed{(10x+3)^2}$$

Factor a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. We can factor a difference of squares by: $a^2 - b^2$ = $(a+b)(a-b)$.

How to Factor a Difference of Squares

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as $(a+b)(a-b)$.

Example 8. Factor the polynomial:

$$\underbrace{49x^2}_{(7x)^2} - \underbrace{16}_{4^2}$$

$$49x^2 - 16 = (7x)^2 - 4^2$$

DIFFERENCE OF SQUARES!

$$= \boxed{(7x+4)(7x-4)}$$

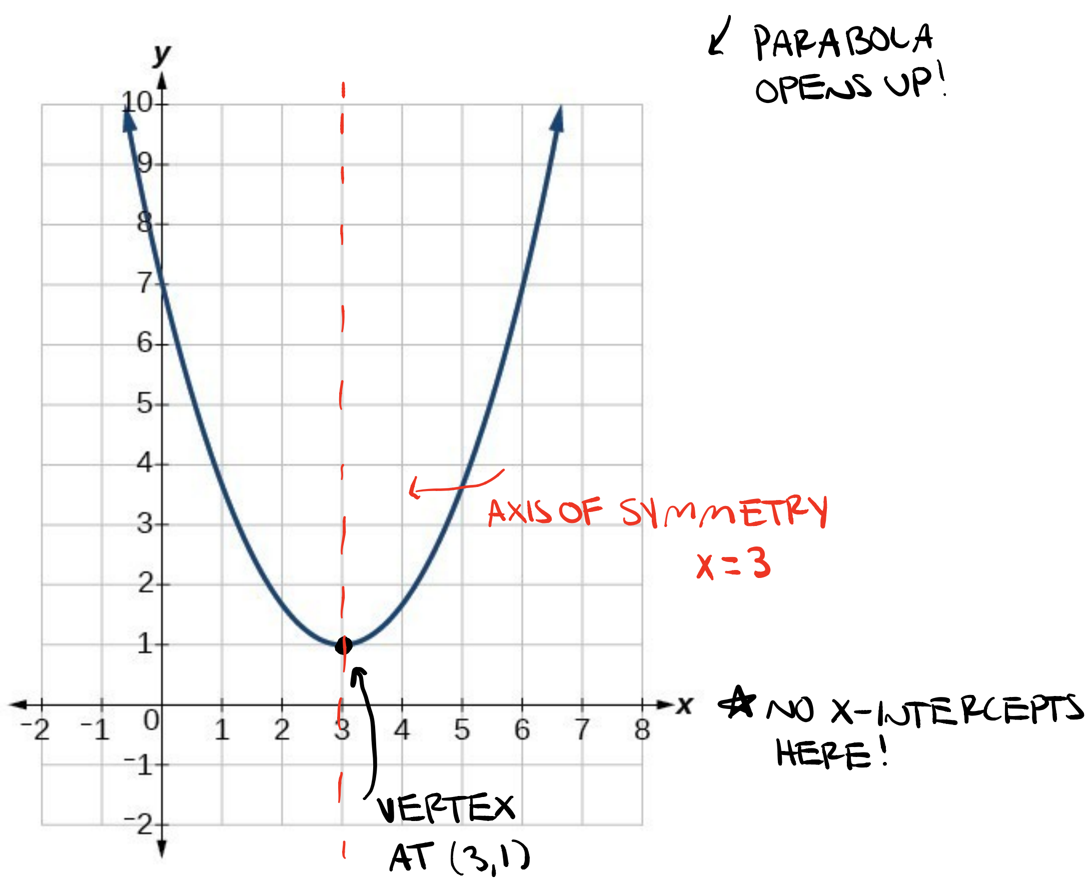
★ $a^2+b^2 \neq (a+b)(a-b)$

DOES NOT EQUAL!

4.2 Graphing Quadratic Functions

Definitions:

The graph of a quadratic function is a U-shaped curve called a PARABOLA. The extreme point of a parabola is called the VERTEX. If the parabola opens up, the vertex represents the lowest point on the graph, called the MINIMUM VALUE of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, called the MAXIMUM VALUE. The graph of a quadratic function is symmetric, with a vertical line drawn through the vertex called the AXIS OF SYMMETRY. The X - INTERCEPTS are the points where the parabola crosses the x-axis.



Parabolas and Quadratic Functions

General Form of a Quadratic Function: The general form of a quadratic function is ax^2+bx+c, where a, b, c are real numbers and $a \neq 0$. If $a>0$, the parabola opens up. If $a<0$, the parabola opens down.

Standard Form of a Quadratic Function: The standard form of a quadratic function is $f(x)=a(x-h)^2+k$, where $a \neq 0$. If $a>0$, the parabola opens up. If $a<0$, the parabola opens down.

Axis of Symmetry: The equation to find the axis of symmetry is given by

$$x = -\frac{b}{2a}. \qquad (\text{OR } x=h)$$

Vertex: The vertex is located at (h, k), where

$$h = -\frac{b}{2a}, \quad \text{and} \quad k = f(h) = f\left(-\frac{b}{2a}\right)$$

How to Find the Equation of a Quadratic Function Given its Graph

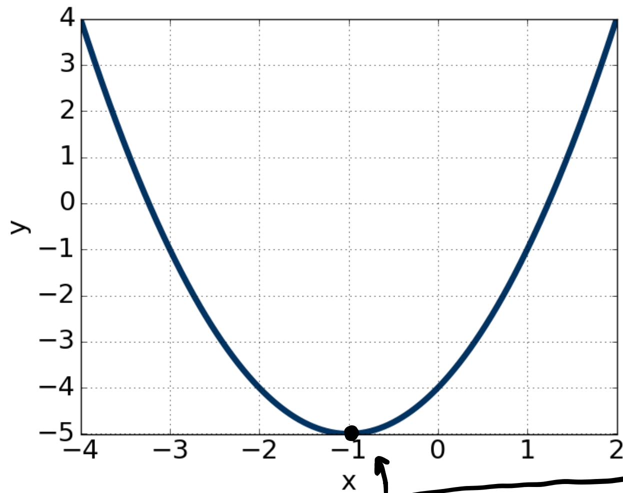
1. Identify the coordinates of the vertex, (h, k).
2. Substitute the values of h and k (found in 1) into the equation $f(x) = a(x-h)^2 + k$.
3. Substitute the values of any other point on the parabola (other than the vertex) for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. Determine if a is positive or negative.

6. Expand and simplify to write in general form.

Example 9. Write the equation of the graph below in the form $ax^2 + bx + c$, assuming $a = 1$ or $a = -1$:

★ PARABOLA OPENS UP, SO $a > 0$

$\Rightarrow \boxed{a = 1}$



VERTEX IS $(-1, -5)$

$\Rightarrow h = -1$

$k = -5$

$f(x) = a(x-h)^2 + k$

$f(x) = 1(x-(-1))^2 + (-5)$

$f(x) = (x+1)^2 - 5$

$f(x) = (x+1)(x+1) - 5$

$f(x) = x^2 + x + x + 1 - 5$

$\boxed{f(x) = x^2 + 2x - 4}$

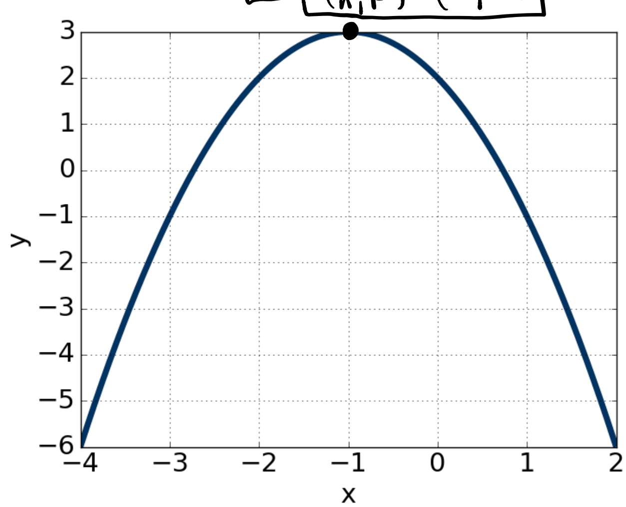
Example 10. Write the equation of the graph below in the form $ax^2 + bx + c$, assuming $a = 1$ or $a = -1$:

VERTEX IS $(h, k) = (-1, 3)$

★ PARABOLA OPENS DOWN!

$\Rightarrow a < 0$

$\Rightarrow \boxed{a = -1}$



$f(x) = a(x-h)^2 + k$

$f(x) = -1(x-(-1))^2 + 3$

$f(x) = -(x+1)^2 + 3$

$f(x) = -(x+1)(x+1) + 3$

$f(x) = -(x^2 + x + x + 1) + 3$

$f(x) = -(x^2 + 2x + 1) + 3$

$f(x) = -x^2 - 2x - 1 + 3$

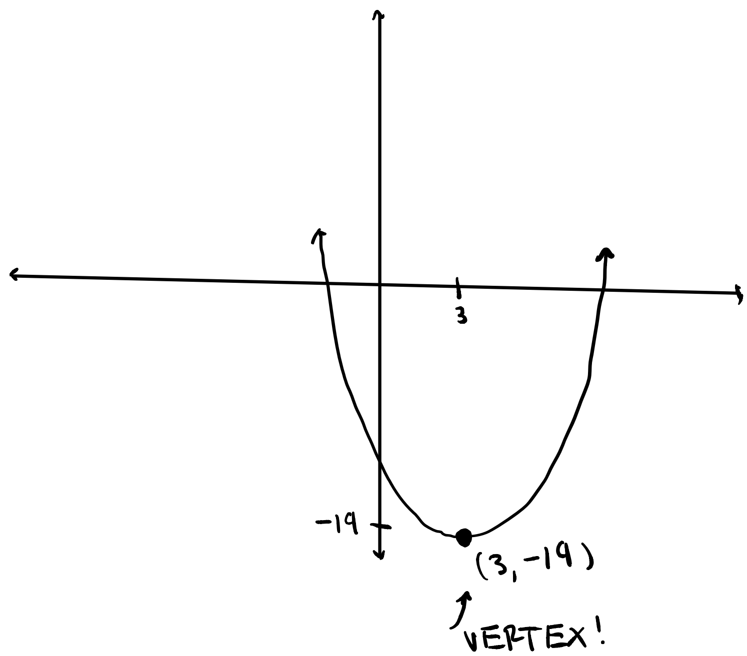
$\boxed{f(x) = -x^2 - 2x + 2}$

Example 11. Graph the equation $f(x) = (x-3)^2 - 19$ ★ CAN USE DESMOS!

★ THIS IS THE FORM $f(x) = a(x-h)^2 + k$

$\Rightarrow$ $h=3$ AND $k=-19$, SO VERTEX IS $(h,k) = (3,-19)$

$\Rightarrow$ $a=1$, SO $a>0$ AND THUS, THE PARABOLA OPENS UP!



★ AXIS OF SYMMETRY IS $x=h$, SO $x=3$!

4.3 Solving Quadratics by Factoring

How to Find the x-Intercept and y-Intercept of a Quadratic Function:

1. To find the y-intercept, evaluate the function at $x=0$.
2. To find the x-intercepts, solve the quadratic equation $f(x)=0$.

Note 6. Solving the quadratic equation $f(x) = 0$ can be done by factoring, or by using the quadratic formula. First, we will solve quadratic equations by factoring. To solve $f(x) = 0$, we will factor $f(x)$ and set each factor equal to 0.

Example 12. Solve the quadratic equation by factoring:

$$15x^2 + 9x - 6 = 0$$

1. FACTOR $15x^2+9x-6$:

$(15)(-6) = -90$

-90	1
90	-1
-45	2
45	-2
-30	3
30	-3
-18	5
18	-5
15	-6
-15	6

$(15)(-6) = -90$
AND
$15 - 6 = 9$!

$\Rightarrow 15x^2 + 9x - 6 = 0$

$\underbrace{15x^2 + 15x}\ \underbrace{-6x - 6} = 0$

$15x(x+1) - 6(x+1) = 0$

$(x+1)(15x-6) = 0$

★ SET EACH FACTOR EQUAL TO 0:

$(x+1)(15x-6)=0$

$x+1=0$
$\boxed{x=-1}$

$15x-6=0$
$15x=6$
$\boxed{x=\frac{6}{15}}$

THESE ARE THE SOLUTIONS TO $f(x)=0$, ie THEY ARE THE ZEROS AND Y-INTERCEPTS

Example 13. Solve the quadratic equation by factoring:

$$4x^2 + 12x + 9 = 0$$

1. FACTOR $4x^2+12x+9$:

$4x^2 = (2x)^2$ AND $9 = 3^2$, SO THIS MIGHT BE A PERFECT SQUARE TRINOMIAL

$2(2x)(3) = (4x)(3) = 12x$ ✓

$\Rightarrow 4x^2+12x+9 = (2x)^2 + 2(2x)(3) + 3^2 = (2x+3)^2$

SO, $4x^2+12x+9=0$

$\Rightarrow (2x+3)^2 = 0$

$\Rightarrow (2x+3)(2x+3) = 0$

$2x+3=0$

$2x=-3$

$\boxed{x = -\frac{3}{2}}$

Example 14. Solve the quadratic equation by factoring:

$$350x^2 + 30x - 8 = 0$$

1. FACTOR $350x^2+30x-8$:

$(350)(-8) = -2800$:

FIND TWO FACTORS OF -2800 WHOSE SUM IS 30: 70 AND -40 WORK!

$(70)(-40) = -2800$ AND $70+(-40) = 30$

$\Rightarrow 350x^2+30x-8=0$

$\Rightarrow 350x^2+70x-40x-8=0$

$\Rightarrow 70x(5x+1) - 8(5x+1) = 0$

$\Rightarrow (5x+1)(70x-8) = 0$

SET EACH FACTOR EQUAL TO 0!

$5x+1=0$

$5x=-1$

$\boxed{x=-\frac{1}{5}}$

$70x-8=0$

$70x=8$

$x = \frac{8}{70}$

$\boxed{x=\frac{4}{35}}$

THESE ARE THE SOLUTIONS/ZEROS/X-INTERCEPTS

4.4 Solving Quadratics using the Quadratic Formula

The Quadratic Formula: $f(x) = ax^2 + bx + c$

To solve the quadratic function $f(x) = 0$, we can use the quadratic formula which is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note 7. Recall that to find the x-intercepts of a quadratic function, we solve the quadratic equation $f(x) = 0$. So, to find the x-intercepts, we can solve by factoring, or we can solve using the quadratic formula. The quadratic formula will always work, but sometimes it is much more tedious to use.

Example 15. Solve the quadratic equation using the quadratic formula:

$$\boxed{x = 1 + \sqrt{3}, \; x = 1 - \sqrt{3}}$$

$$4x^2 - 8x - 8 = 0$$

$a = 4$, $b = -8$, $c = -8$

$$x = -\frac{b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(-8)}}{2(4)} = \frac{8 \pm \sqrt{64+128}}{8} = \frac{8 \pm \sqrt{192}}{8} = \frac{8 \pm \sqrt{64 \cdot 3}}{8}$$

$$= \frac{8 \pm 8\sqrt{3}}{8} = \frac{8}{8} \pm \frac{8\sqrt{3}}{8} = \boxed{1 \pm \sqrt{3}}$$

Example 16. Solve the quadratic equation using the quadratic formula:

$$2x^2 - 8x + 7 = 0$$

$$x = -\frac{b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(2)(7)}}{2(2)} = \frac{8 \pm \sqrt{64-56}}{4}$$

$$= \frac{8 \pm \sqrt{8}}{4} = \frac{8 \pm \sqrt{4 \cdot 2}}{4} = \frac{8 \pm 2\sqrt{2}}{4} = \frac{8}{4} \pm \frac{2\sqrt{2}}{4} = \boxed{2 \pm \frac{\sqrt{2}}{2}}$$