Module 5 Lecture Notes

MAC1105

Summer B 2019

5 Radical Functions

5.1 Domain

Definition

A RELATION is a correspondence between two sets A and B. A relation is expressed as a pair of ordered pairs (x, y), where x is an element of set A and y is an element of set B.

Definition

A FUNCTION from a set A into a set B is a relation that assigns to each element x in set A EXACTLY ONE element y in the set B.

Note 1. The DOMAIN of a relation is the set of all first elements and the RANGE is the set of all second elements in the ordered pairs.

Example 1. Consider the following examples:

1. $\{(1,2),(3,5),(2,97)\}$ IS A FUNCTION BECAUSE EACH "X" HAS ONLY ONE "Y". THE DOMAIN IS $\{1,3,2\}$. THE RANGE IS $\{2,5,97\}$.

2. $\{(1,2),(1,12),(-12,13),(9,17)\}$ IS NOT A FUNCTION BECAUSE 1 CORRESPONDS TO TWO DIFFERENT Y-VALUES

Definition

The DOMAIN of a function $y = f(x)$ is the set of all real numbers x for which the expression is defined.

The Standard Form of a Radical Function

The standard form of a radical function is given by $f(x) = a\sqrt[n]{bx-c} + k$.

Note 2. For now, we will write the standard form of a radical function as $f(x) = a\sqrt{x-h} + k$.

Observe that when we set $bx - c = 0$ and solve for x we get : $bx = c$

$x = \frac{c}{b}$

The Standard Form of a Radical Function

Standard Form: The standard form of a radical function is given by $f(x) = a\sqrt{x-h} + k$.

Vertex: The vertex of a radical function is (h, k).

Note 3. In our formula, a tells us how wide our graph will be. It is the "stretch factor" of the graph. n tells us what root we are taking.

Question 1: Can we take the square root of a negative number?

NO, ONLY WITH COMPLEX NUMBERS

☆ $x - h \geq 0$

$x \geq h$

☆ $bx - c \geq 0$

$x \geq \frac{c}{b}$

Question 2: Can we take the cube root of a negative number?

YES! Ex: $\sqrt[3]{-8} = -2$ BECAUSE $(-2)^3 = (-2)(-2)(-2)$

$= (4)(-2)$

$= -8$

Question 3: Can we take the even root of a negative number?

NO - ONLY WITH COMPLEX NUMBERS

Question 4: Can we take the odd root of a negative number?

YES!

Note 4. From question 2 and 4, we can see that the domain of a radical function with an odd root (when n is odd) is $(-\infty, \infty)$. From question 2 and 3, we can see that there are two possibilities for the domain of an even root function:

$f(x) = a\sqrt[n]{bx-c} + k$

$bx - c \geq 0$

$bx \geq c$

$x \geq \frac{c}{b}$

DOMAIN: $\left(-\infty, \frac{c}{b}\right]$

DOMAIN: $\left[\frac{c}{b}, \infty\right)$

Ex: $f(x) = \sqrt{-2x+5}$

$-2x + 5 \geq 0$

$-2x \geq -5$

$x \leq 5/2$

$(-\infty, 5/2]$

Ex: $f(x) = \sqrt{2x+5}$

$2x - 5 \geq 0$

$2x \geq 5$

$x \geq \frac{5}{2}$

$[5/2, \infty)$

☆ TO FIND THE DOMAIN OF AN EVEN ROOT FUNCTION, SET THE "STUFF" UNDER THE RADICAL ≥ 0 AND SOLVE

Example 2. Write the domain of the function in interval notation:

$$\sqrt{x-2}$$

★ EVEN ROOT ⇒ RESTRICTED DOMAIN

$x - 2 \geq 0$

$x \geq 2$

$\boxed{[2, \infty)}$

Example 3. Write the domain of the function in interval notation:

$$\sqrt[3]{-8x+6}$$

★ ODD ROOT ⇒ DOMAIN IS NOT RESTRICTED!

$\Rightarrow \boxed{(-\infty, \infty)}$

Example 4. Write the domain of the function in interval notation:

$$\sqrt[8]{5x+5}$$

★ EVEN ROOT ⇒ RESTRICTED DOMAIN!

$5x+5 \geq 0$

$5x \geq -5$

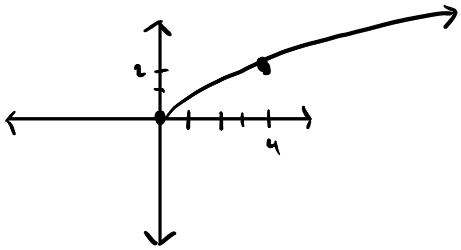
$x \geq -\frac{5}{5}$

$x \geq -1$

$\boxed{[-1, \infty)}$

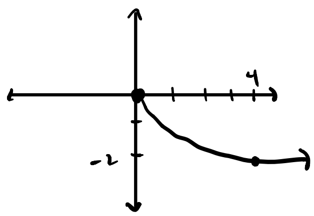
5.2 Graphing Radical Functions

The graph for $\sqrt{x}$ looks like:



x	y
0	0
4	2
16	4

The graph for $-\sqrt{x}$ looks like:



x	y
0	0
4	-2
16	-4

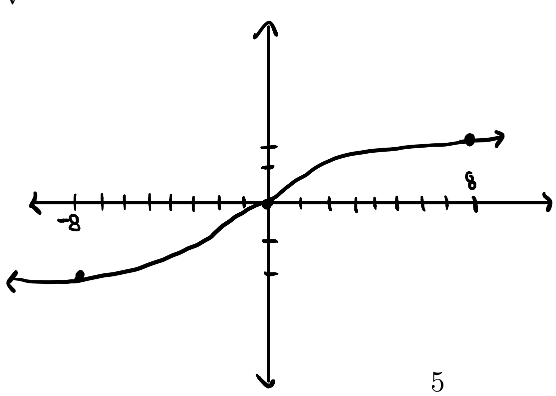
Note 5. Observe that the graph of $-\sqrt{x}$ is the reflection about the x - AXIS of the graph of $\sqrt{x}$.

Reflections Across an Axis

The graph $y = -f(x)$ is the reflection about the X - AXIS of the graph of $y = f(x)$. The graph of $y = f(-x)$ is the reflection about the Y - AXIS of the graph of $y = f(x)$.

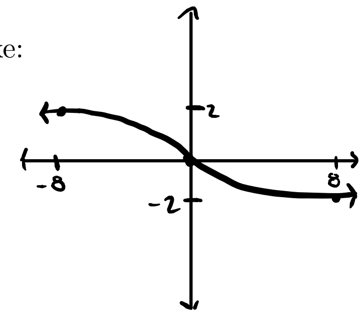
Note 6. In fact, this is what the graph for any *even* root function looks like.

The graph for $\sqrt[3]{x}$ looks like:



x	y
-8	2
0	0
8	2

The graph for $-\sqrt[3]{x}$ looks like:



GRAPH OF $\sqrt[3]{x}$ FLIPPED ACCROSS THE X-AXIS

Note 7. In fact, this is what the graph for any *odd* root function looks like.

Finding the Equation of a Radical Function Given its Graph

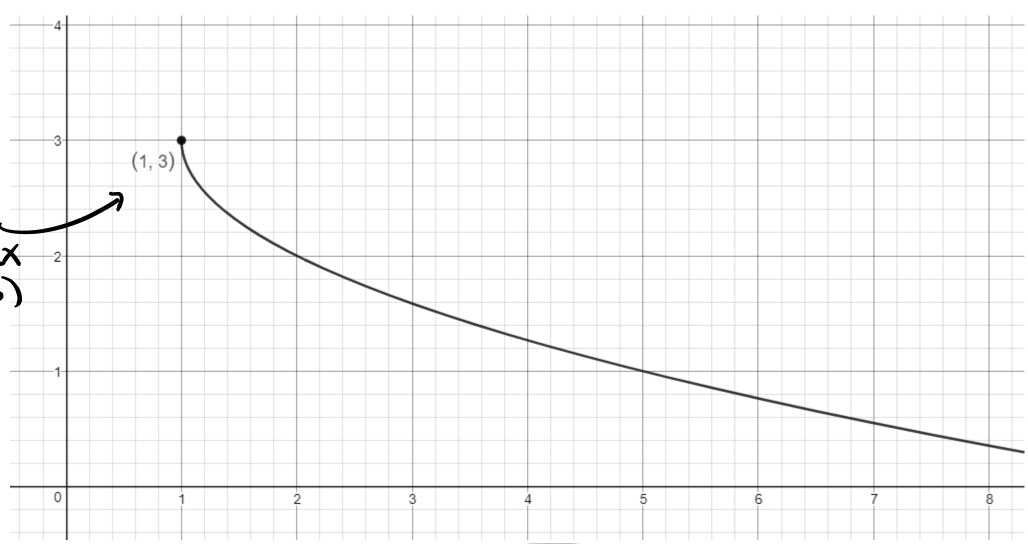
1. Determine whether the root of the function is odd or even.
2. Determine whether a is greater than 0 or less than 0.
3. Find the coordinates for the vertex of the function.
4. Remove any decimals under the radical sign by setting the expression under the radical equal to 0.

Example 5. Write the equation of the following function:

* EVEN ROOT FUNCITION!

* $a < 0$

VERTEX IS (1,3)



$f(x) = a\sqrt{x-h} + k$

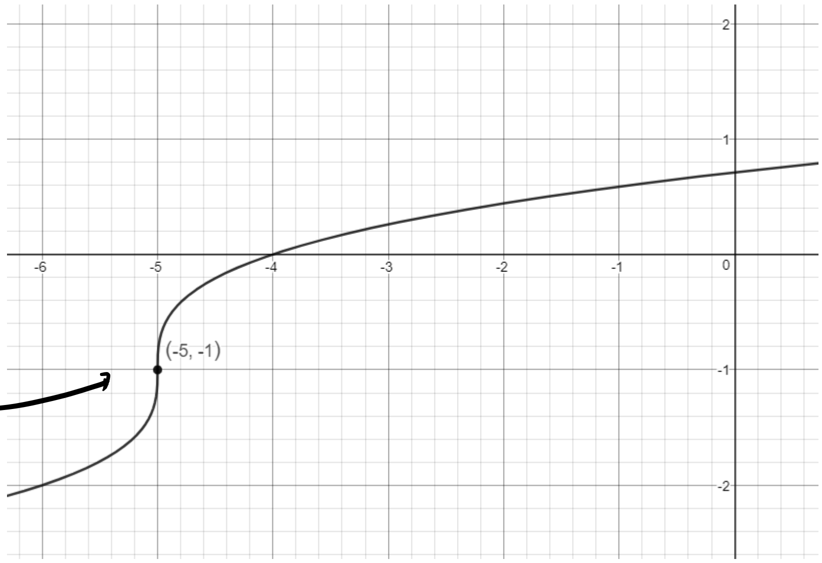
$f(x) = -1\sqrt{x-1} + 3$

$\boxed{f(x) = -\sqrt{x-1} + 3}$

Example 6. Write the equation of the following function:

★ODD ROOT FUNCTION!

★ $a > 0$



VERTEX IS $(-5,-1)$

$f(x) = a\sqrt[3]{x-h} + k$

$f(x) = 1\sqrt[3]{x-(-5)} + (-1)$

$\boxed{f(x) = \sqrt[3]{x+5} - 1}$

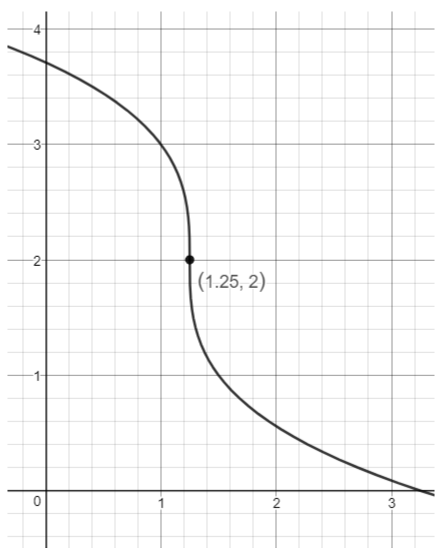
Example 7. Write the equation of the following function:

★ ODD ROOT FUNCTION

★ $a < 0$

★ VERTEX IS $(h,k) = (1.25, 2)$

IN FRACTION FORM, THIS IS $\left(\frac{5}{4}, 2\right)$



$f(x) = a\sqrt[3]{x-h} + k$

$f(x) = -1\sqrt[3]{x-\frac{5}{4}} + 2$

$f(x) = -\sqrt[3]{x-\frac{5}{4}} + 2$

$\left(x - \frac{5}{4} = 0\right)4$

$4x - 5 = 0$

$f(x) = -\sqrt[3]{x-\frac{5}{4}} + 2$

$\boxed{f(x) = -\sqrt[3]{4x-5} + 2}$

5.3 Solving Radical Equations (Linear)

Rational Exponents

A rational exponent indicates a power in the numerator and a root in the denominator. We can write rational exponents in many different ways:

$$(a)^{m/n} = \left(a^{1/n}\right)^m = \underline{a^{m/n}} = \sqrt[n]{a^m} = \underline{(\sqrt[n]{a})^m}$$

Example 8. We can write $2^{1/2}$ and $4^{2/3}$ as follows:

- $2^{1/2} = \sqrt[2]{2}$
- $4^{2/3} = \sqrt[3]{4^2} = \sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = 2\sqrt[3]{2}$

Example 9. Evaluate $8^{2/3}$

$$8^{2/3} = \sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 2^2 = 4$$

Example 10. Evaluate $64^{-1/3}$

$$64^{-1/3} = \frac{1}{64^{1/3}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$$

Definition

A radical equation is an equation that contains variables in the RADICAND (expression under the radical).

Note 8. When solving radical equations, we need to be careful of finding solutions that are not actually real solutions to our function.

Definition

An EXTRANEOUS SOLUTION is a root of an equation that is not actually a real solution to the equation.

Note 9. We can "get rid of" a radical as follows:

- $\sqrt{2}$: $(\sqrt{2})^2 = (2^{1/2})^2 = 2^{2/2} = 2^1 = 2$
- $\sqrt[n]{x}$: $(\sqrt[n]{x})^n = (x^{1/n})^n = x^{n/n} = x^1 = x$

How to Solve a Radical Equation

1. Isolate the radical expression on one side of the equation. Put all remaining terms on the other side.
2. For a square root radical, raise both sides to the 2nd power. Doing so eliminates the radical.
3. Solve the remaining equation.
4. If there is still a radical symbol, repeat steps 1-2.
5. CHECK YOUR SOLUTIONS by substituting into the original equation.

Note 10. If we have an nth root radical, raise both sides to the nth power in step 2 above.

- $(\sqrt[n]{70})^n = (70^{1/n})^n = 70^{n/n} = 70$
- $(\sqrt[6]{7})^6 = (7^{1/6})^6 = 7^{6/6} = 7$

Example 11. Solve the following equation:

$$\sqrt{3x-3} = \sqrt{7x-2}$$

$$\left(\sqrt{3x-3}\right)^2 = \left(\sqrt{7x-2}\right)^2$$

SQUARE BOTH SIDES TO GET RID OF THE RADICAL

$$3x-3 = 7x-2$$

$$-3x \quad = -3x$$

$$-3 = 4x-2$$

$$+2 = \quad +2$$

$$-1 = 4x$$

$$\boxed{-\frac{1}{4} = x}$$

CHECK SOLUTION!

$$\sqrt{3\left(-\frac{1}{4}\right)-3} \overset{?}{=} \sqrt{7\left(-\frac{1}{4}\right)-2}$$

$$\sqrt{-\frac{3}{4}-3} \overset{?}{=} \sqrt{-\frac{7}{4}-2}$$

$$\sqrt{-\frac{3}{4}-\frac{12}{4}} \overset{?}{=} \sqrt{-\frac{7}{4}-\frac{8}{4}}$$

$$\sqrt{-\frac{15}{4}} \overset{?}{=} \sqrt{-\frac{15}{4}}$$

* CAN'T TAKE THE SQUARE ROOT OF A NEGATIVE NUMBER

$\Rightarrow$ $\boxed{\text{NO REAL SOLUTIONS}}$

Example 12. Solve the following equation:

$$\sqrt{3x+8} = \sqrt{7x-2}$$

$$\left(\sqrt{3x+8}\right)^2 = \left(\sqrt{7x-2}\right)^2$$

$$3x+8 = 7x-2$$

$$-3x = -3x$$

$$8 = 4x-2$$

$$+2 = +2$$

$$10 = 4x$$

$$\frac{10}{4} = x \Rightarrow \boxed{\frac{5}{2} = x}$$

CHECK FOR EXTRANEOUS SOLUTION

$$\sqrt{3\left(\frac{5}{2}\right)+8} \stackrel{?}{=} \sqrt{7\left(\frac{5}{2}\right)-2}$$

$$\sqrt{\frac{15}{2}+8} \stackrel{?}{=} \sqrt{\frac{35}{2}-2}$$

$$\sqrt{\frac{15}{2}+\frac{16}{2}} \stackrel{?}{=} \sqrt{\frac{35}{2}-\frac{4}{2}}$$

$$\sqrt{\frac{31}{2}} = \sqrt{\frac{31}{2}}$$ ✓ YES, IT WORKS

$\Rightarrow$ SOLUTION IS $\boxed{x = \frac{5}{2}}$

5.4 Solving Radical Equations (Quadratic)

Note 11. Note that solving radical equations that lead to quadratic equations will have 0, 1, or 2 solutions. Follow the same steps as solving radical equations that lead to linear equations.

Example 13. Solve the following equation:

$$\sqrt{20x^2+15}-\sqrt{37x}=0$$

$$+\sqrt{37x}=+\sqrt{37x}$$

$$\sqrt{20x^2+15}=\sqrt{37x}$$

$$\left(\sqrt{20x^2+15}\right)^2=\left(\sqrt{37x}\right)^2$$

$$20x^2+15=37x$$

$$-37x=-37x$$

$$20x^2-37x+15=0$$

$(20)(15)=300$

$(-25)(-12)=300$

AND

$-25+(-12)=-37$

$$\underbrace{20x^2-25x}-\underbrace{12x+15}=0$$

$$5x(4x-5)-3(4x-5)=0$$

$$\underbrace{(4x-5)}\underbrace{(5x-3)}=0$$

$$4x-5=0 \qquad 5x-3=0$$

$$4x=5 \qquad 5x=3$$

$$\boxed{x=\frac{5}{4}} \qquad \boxed{x=\frac{3}{5}}$$

CHECK!

$$\sqrt{20(1.25)^2+15}-\sqrt{37(1.25)}\overset{?}{=}0$$

$$\sqrt{46.25}-\sqrt{46.25}\overset{?}{=}0$$

$$0=0\ \checkmark$$

YES!

$$\Rightarrow \boxed{x=\frac{5}{4}=1.25}$$

$$\sqrt{20(0.6)^2+15}-\sqrt{37(0.6)}\overset{?}{=}0$$

$$\sqrt{22.2}-\sqrt{22.2}\overset{?}{=}0$$

$$0=0\ \checkmark$$

$$\boxed{x=\frac{3}{5}=0.6}$$

Example 14. Solve the following equation:

$$\sqrt{-30x^2-25}-\sqrt{-55x}=0$$

$(6)(5)=30$

★ $(-6)(-5)=30$

AND

$-6+(-5)=-11$

$$+\sqrt{-55x}=+\sqrt{-55x}$$

$$\sqrt{-30x^2-25}=\sqrt{-55x}$$

$$\left(\sqrt{-30x^2-25}\right)^2=\left(\sqrt{-55x}\right)^2$$

$$-30x^2-25=-55x$$

$$0=30x^2-55x+25$$

$$0=5(6x^2-11x+5)$$

$$0=5[6x^2-6x-5x+5]$$

$$0=5[6x(x-1)-5(x-1)]$$

$$0=5\underbrace{(6x-5)}\underbrace{(x-1)}$$

$6x-5=0$ $\quad$ $x-1=0$

$x=\frac{5}{6}$ $\quad$ $x=1$

$x=0.8\overline{3}$

CHECK!

$x=1$

$$\sqrt{-30(1)^2-25}-\sqrt{-55(1)}\stackrel{?}{=}0$$

$$\sqrt{-30-25}-\sqrt{-55}\stackrel{?}{=}0$$

★ CANNOT TAKE SQUARE ROOT OF A NEGATIVE, SO $x=1$ IS NOT A SOLUTION!

$x=0.8\overline{3}$

$$\sqrt{-30(0.8\overline{3})^2-25}-\sqrt{-55(0.8\overline{3})}\stackrel{?}{=}0$$

$$\sqrt{-45.8\overline{3}}-\sqrt{-45.8\overline{3}}\stackrel{?}{=}0$$

★ CANNOT TAKE SQUARE ROOT OF A NEGATIVE $\Rightarrow$ $x=0.8\overline{3}$ NOT A SOLUTION

$\Rightarrow$ $\boxed{\text{NO SOLUTIONS}}$