Module 6 Lecture Notes

MAC1105

Fall 2019

6 Polynomial Functions

6.1 End and Zero Behavior

Note 1. A polynomial of degree 2 or more has a graph with no sharp turns or cusps.

Note 2. The domain of a polynomial function is ____________________.

Definition

The values of x for which $f(x) = 0$ are called the ____________________ or x-intercepts of f.

Note 3. If a polynomial can be factored, we can set each factor equal to zero to find the x-intercepts (or zeros) of the function. Recall that the x-intercepts of a function are where $f(x) = 0$, or $y = 0$. The y-intercepts are where $x = 0$.

How to Find the x-Intercepts of a Polynomial Function, f, by Factoring

1. Set ____________________.

2. If the polynomial function is not in factored form, then factor the polynomial.

3. Set each factor equal to ______ to find the x-intercepts.

Example 1. Find the x and y-intercepts of:

$$g(x) = (x-2)^2(2x+3)$$

Note 4. The graphs of polynomials behave differently at various x-intercepts. Sometimes, a graph will ____________________ the horizontal x-axis at the x-intercepts, and other times the graph will ____________________ or bounce off the horizontal x-axis at the x-intercepts.

Definition

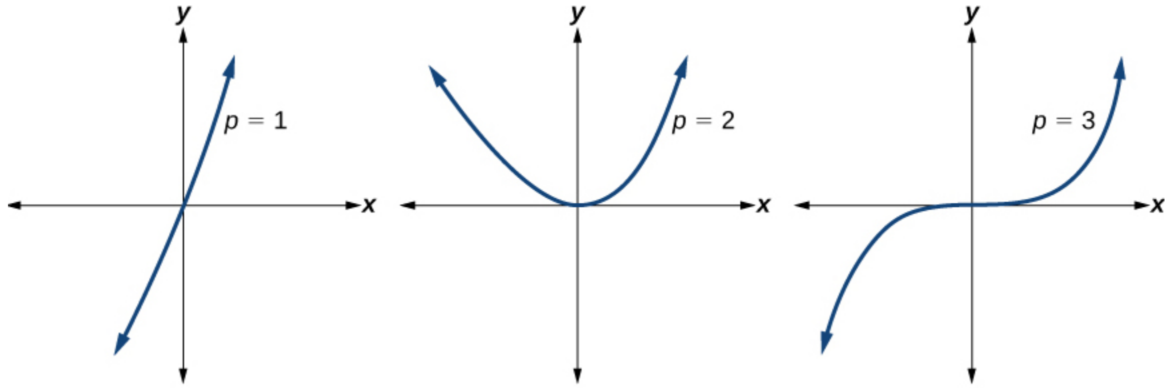
The number of times a given factor appears in the factored form of a polynomial is called the ____________________.

Example 2. From the above example, $g(x) = (x-2)^2(2x+3)$, the factor associated to the zero at $x = 2$ has multiplicity ______. This zero has even multiplicity. The factor associated to the zero at $x = -\dfrac{3}{2}$ has multiplicity ______. This zero has odd multiplicity.

Graphical Behavior of Polynomials at x-Intercepts (Zeros)

If a polynomial contains a factor in the form $(x-h)^p$, the behavior near the x-intercept h is determined by the power p. We say that $x = h$ is a zero of ____________________ p. The graph of a polynomial function will touch the x-axis at zeros with ____________________ multiplicities. The graph of a polynomial function will cross the x-axis at zeros with ____________________ multiplicities. The sum of the multiplicities is the ____________________ of the polynomial function.

Example 3. The graphs below exemplify the behavior of polynomials at their zeros with different multiplicities:



Note 5. The graph of a polynomial function of the form

$$f(x) = a_0 + a_1x + ... + a_{n-1}x^{n-1} + a_nx^n$$

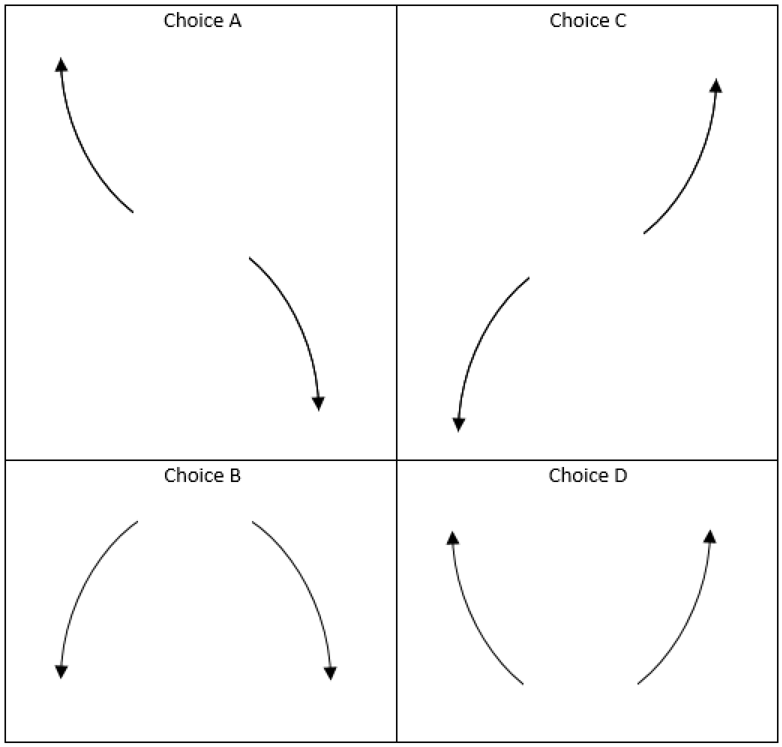
will either __________ or __________ as x increases without bound and will either __________ or __________ as x decreases without bound. This is called the __________ ____________________ of a function.

Example 4. The chart below illustrates the end behavior of a polynomial function:

Even Degree	Odd Degree
Positive Leading Coefficient $a_n > 0$ End Behavior: $x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to \infty$	Positive Leading Coefficient $a_n > 0$ End Behavior: $x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to -\infty$
Negative Leading Coefficient $a_n < 0$ End Behavior: $x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to -\infty$	Negative Leading Coefficient $a_n < 0$ End Behavior: $x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to \infty$

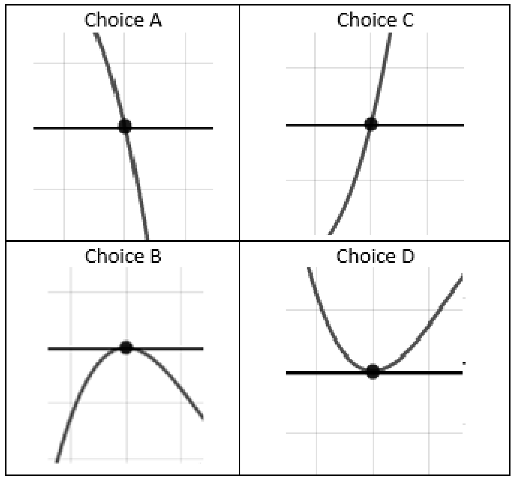
Example 5. Choose the end behavior of the polynomial function:

$$f(x) = -(x+7)^6(x+5)^4(x-5)^3(x-7)^3$$



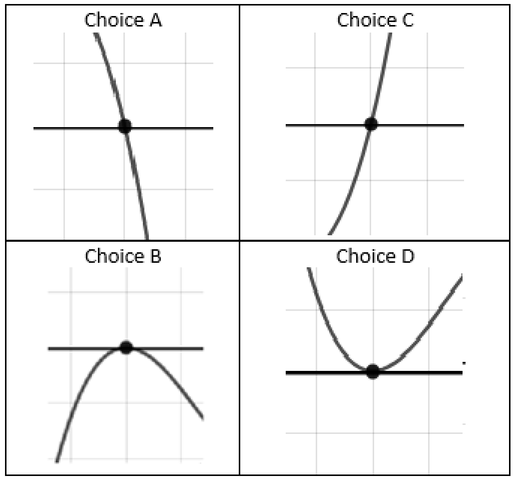
Example 6. Choose the option below that describes the behavior at $x = -3$ of the polynomial:

$$f(x) = (x+6)(x+3)^4(x-3)^3(x-6)$$



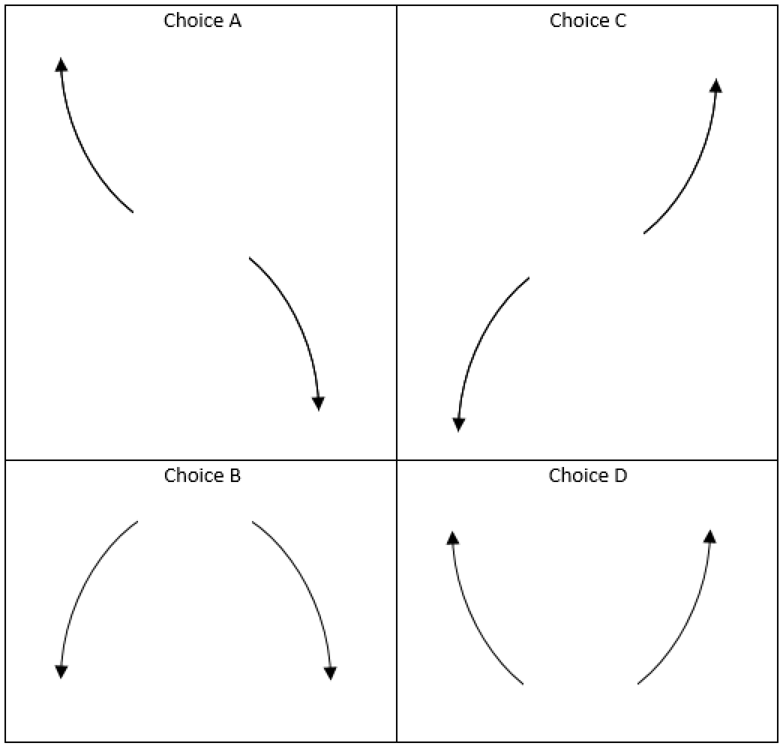
Example 7. Choose the option below that describes the behavior at $x = -9$ of the polynomial:

$$f(x) = (x+9)^3(x+3)^5(x-3)^3(x-9)^2$$



Example 8. Choose the end behavior of the polynomial function:

$$f(x) = (x+9)(x+4)^6(x-4)^3(x-9)$$



6.2 Graphing Polynomials

Definition

A ____________ ____________ of the graph of a polynomial function is the point where a function changes from rising to falling or from falling to rising. A polynomial of degree n will have at most ____________ turning points.

How to Determine the Zeros and Multiplicities of a Polynomial of Degree n Given its Graph

1. If the graph crosses the x-axis at the intercept, it is a zero with ____________ ____________.
2. If the graph touches the x-axis and bounces off the axis, it is a zero with ____________ ____________.
3. The sum of the multiplicities is _____.

Definition

If a polynomial of lowest degree p has x-intercepts at $x = x_1, x_2, ..., x_n$, then the polynomial can be written in factored form:

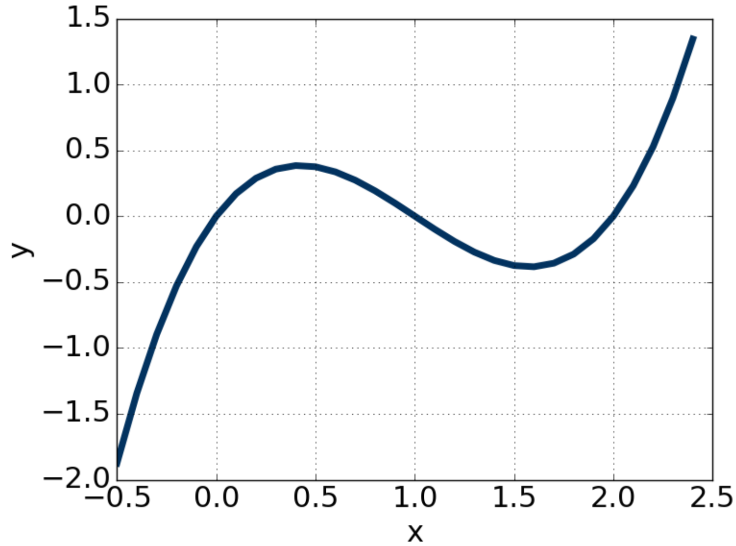
____.

Note 6. In the factored form of a polynomial, the powers on each factor can be determined by the behavior of the graph at the corresponding ____________________, and the stretch factor a can be determined given a value of the function other than the _____ - ____________________.

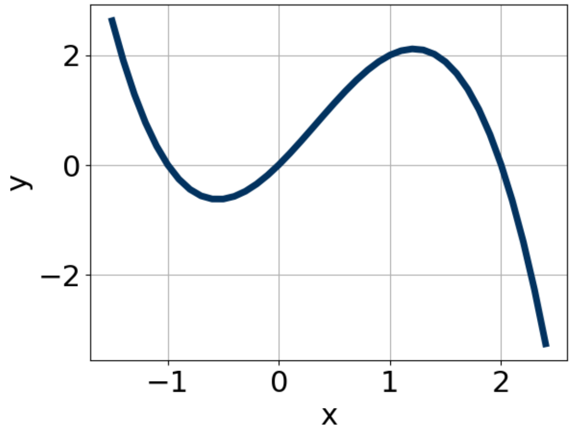
How to Determine a Polynomial Function Given its Graph

1. Identify the _____ - ____________________ to determine the factors of the polynomial.
2. Examine the behavior of the graph at x-intercepts to determine the ____________________ of each factor.
3. Find the polynomial of least degree containing all the factors found in step 2.
4. Use any other point on the graph (typically the y-intercept) to determine the stretch factor (or, you can analyze the end behavior of the graph to determine the stretch factor).

Example 9. Write an equation of the function graphed below:



Example 10. Write an equation of the function graphed below:



How to Sketch the Graph of a Polynomial Function

1. Find the x-intercepts (zeros).
2. Find the y-intercepts.
3. Check for symmetry. If the function is an even function, then its graph is symmetric about the ______ - ________________ (that is, $f(-x) = f(x)$). If the function is an odd function, then its graph is symmetric about the ______ - ________________ (that is, $f(-x) = -f(x)$).
4. Determine the behavior of the polynomial at the zeros using their ______________________.
5. Determine the ________________ ______________________.
6. Sketch a graph.
7. Check that the number of ______________________ ________________ does not exceed one less than the degree of the polynomial.

6.3 Lowest Degree Polynomial

The Factor Theorem

k is a zero of $f(x)$ if and only if ________________ is a factor of $f(x)$.

Note 7. The following statements are equivalent:

Note 8. If we are given the zeros of a polynomial, we can use the ________________ ________________ to construct the lowest-degree polynomial.

Example 11. Construct the lowest-degree polynomial given the zeros below:

$$3, -3, -4$$

Example 12. Construct the lowest-degree polynomial given the zeros below:

$$-\frac{4}{3}, -\frac{3}{2}, -3$$

Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n > 0$, then $f(x)$ has at least one ____________________ ____________________. In fact, if $f(x)$ is a polynomial of degree $n > 0$ and a is a nonzero real number, then $f(x)$ has exactly n ____________________ ____________________:

____

where $c_1, c_2, ..., c_n$ are complex numbers. That is, $f(x)$ has _____ ______________ if we allow for multiplicities.

Note 9. This does NOT mean that every polynomial has an imaginary zero. Real numbers are a subset of the complex numbers, but complex numbers are not a subset of the real numbers.

The Linear Factorization Theorem

If f is a polynomial function of degree n, then f has n ________________, and each factor is of the form ________________, where c is a complex number. That is, a polynomial function has the same number of linear factors as its degree.

Complex Conjugate Theorem

Suppose f is a polynomial function with real coefficients. If f has a complex zero of the form $a+bi$, then the ____________________ ____________________ of the zero, $a-bi$, is also a zero.

A Closer Look at the Zeros of a Polynomial Function

Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Case 1: $b^2 - 4ac$ is positive and not a perfect square:

$$x = \underline{\hspace{3cm}} \pm \underline{\hspace{3cm}}$$

$$x = \underline{\hspace{4.5cm}} \pm \underline{\hspace{4.5cm}}$$

Case 2: $b^2 - 4ac$ is negative:

$$x = \underline{\hspace{3cm}} \pm \underline{\hspace{3cm}}$$

$$x = \underline{\hspace{4.5cm}} \pm \underline{\hspace{4.5cm}}$$

Example 13. Construct the lowest-degree polynomial given the zeros below:

$$\sqrt{2}, \frac{1}{3}$$

Example 14. Construct the lowest-degree polynomial given the zeros below:

$$4 + 3i, -\frac{2}{5}$$