Module 6 Lecture Notes

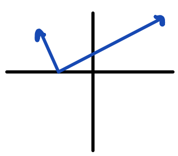
MAC1105

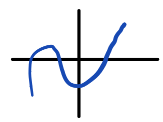
Fall 2019

6 Polynomial Functions

6.1 End and Zero Behavior

Note 1. A polynomial of degree 2 or more has a graph with no sharp turns or cusps.



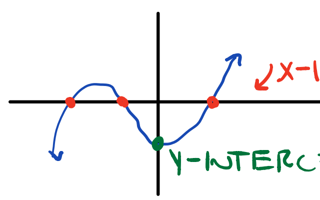


Note 2. The domain of a polynomial function is $(-\infty, \infty)$.

Definition

The values of x for which $f(x) = 0$ are called the ZEROS or x-intercepts of f.

Note 3. If a polynomial can be factored, we can set each factor equal to zero to find the x-intercepts (or zeros) of the function. Recall that the x-intercepts of a function are where $f(x) = 0$, or $y = 0$. The y-intercepts are where $x = 0$.



How to Find the x-Intercepts of a Polynomial Function, f, by Factoring

1. Set $f(x) = 0$.
2. If the polynomial function is not in factored form, then factor the polynomial.

3. Set each factor equal to 0 to find the x-intercepts.

Example 1. Find the x and y-intercepts of:

THIS ZERO HAS "MULTIPLICITY" 2 BECAUSE $(x-2)^2$

$$g(x) = (x-2)^2(2x+3)$$

• X-INTERCEPTS: $x-2=0$, $x=2$ AND $2x+3=0$, $x=-\frac{3}{2}$ $\Rightarrow$ X-INTERCEPTS ARE $(2,0)$ AND $\left(-\frac{3}{2},0\right)$

• Y-INTERCEPT: $g(0)=(0-2)^2(2(0)+3)=(-2)^2(3)=4\cdot 3=12$ Y-INTERCEPT IS $(0,12)$

Note 4. The graphs of polynomials behave differently at various x-intercepts. Sometimes, a graph will CROSS the horizontal x-axis at the x-intercepts, and other times the graph will TOUCH or bounce off the horizontal x-axis at the x-intercepts.

Definition

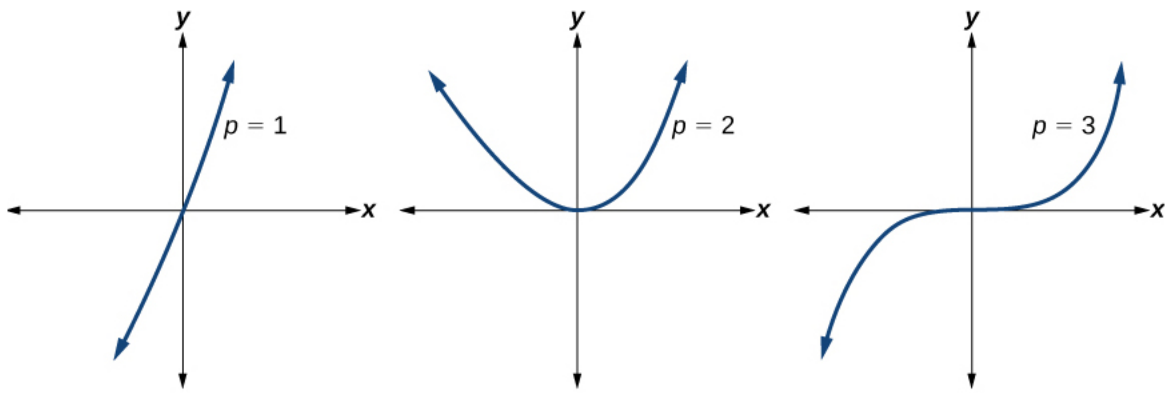
The number of times a given factor appears in the factored form of a polynomial is called the MULTIPLICITY.

Example 2. From the above example, $g(x) = (x-2)^2(2x+3)$, the factor associated to the zero at $x = 2$ has multiplicity 2. This zero has even multiplicity. The factor associated to the zero at $x = -\frac{3}{2}$ has multiplicity 1. This zero has odd multiplicity.

Graphical Behavior of Polynomials at x-Intercepts (Zeros)

If a polynomial contains a factor in the form $(x-h)^p$, the behavior near the x-intercept h is determined by the power p. We say that $x = h$ is a zero of MULTIPLICITY p. The graph of a polynomial function will touch the x-axis at zeros with EVEN multiplicities. The graph of a polynomial function will cross the x-axis at zeros with ODD multiplicities. The sum of the multiplicities is the DEGREE of the polynomial function.

Example 3. The graphs below exemplify the behavior of polynomials at their zeros with different multiplicities:



ODD MULTIPLICITY
⇒ CROSSES X-AXIS
AT THE ZERO

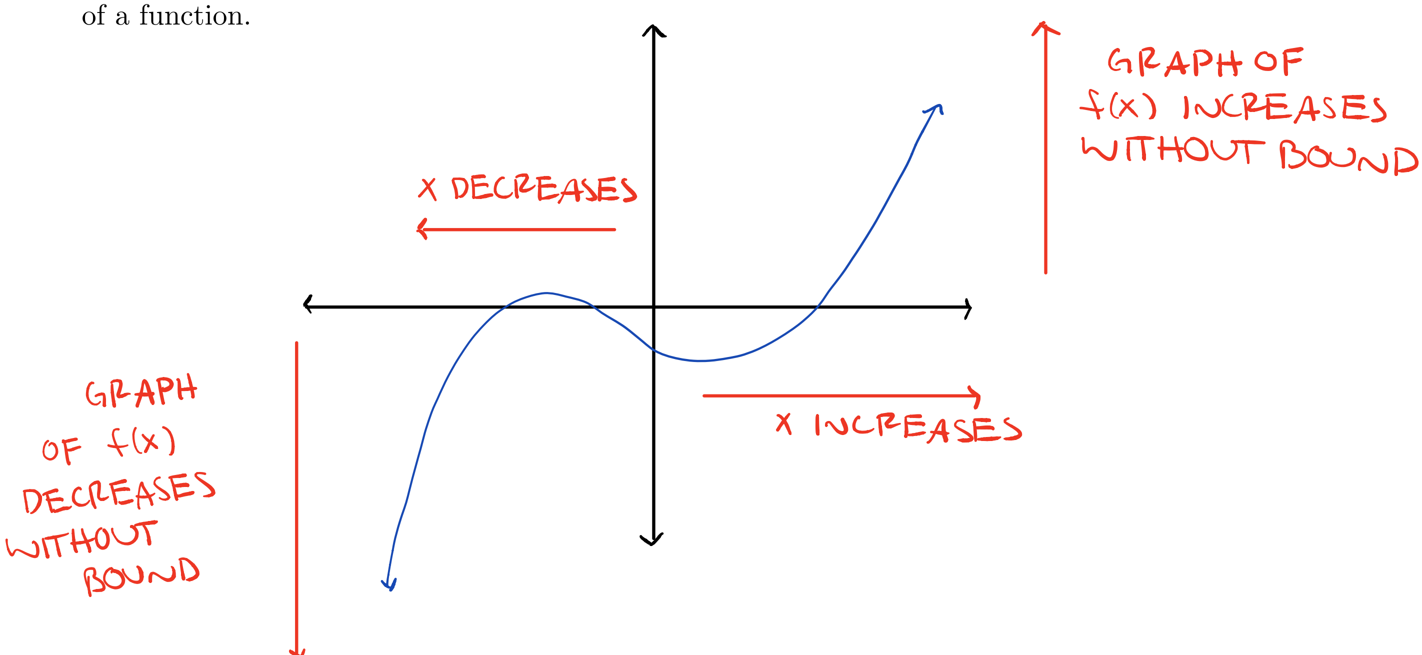
EVEN
MULTIPLICITY
⇒ GRAPH TOUCHES
X-AXIS AT THE ZERO
AND BOUNCES OFF

ODD MULTIPLICITY
⇒ CROSSES

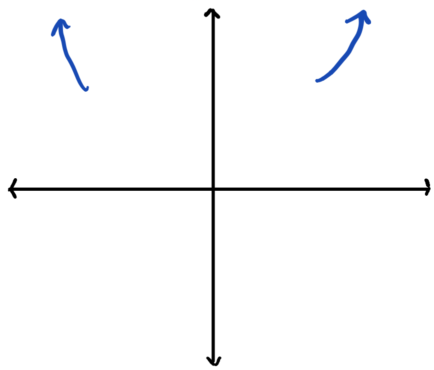
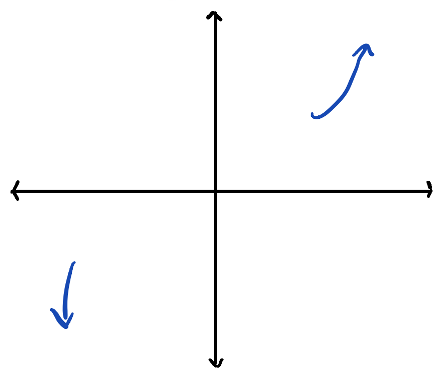
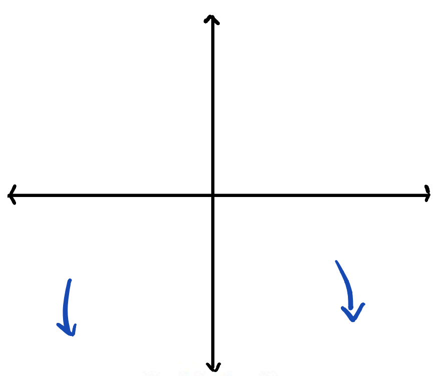
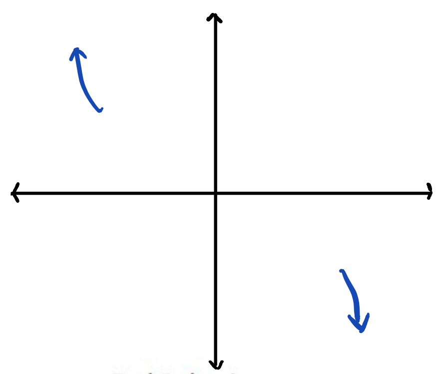
Note 5. The graph of a polynomial function of the form

$$f(x) = a_0 + a_1 x + ... + a_{n-1}x^{n-1} + a_n x^n$$

will either _RISE_ or _FALL_ as x increases without bound and will either _RISE_ or _FALL_ as x decreases without bound. This is called the _END_ _BEHAVIOR_ of a function.

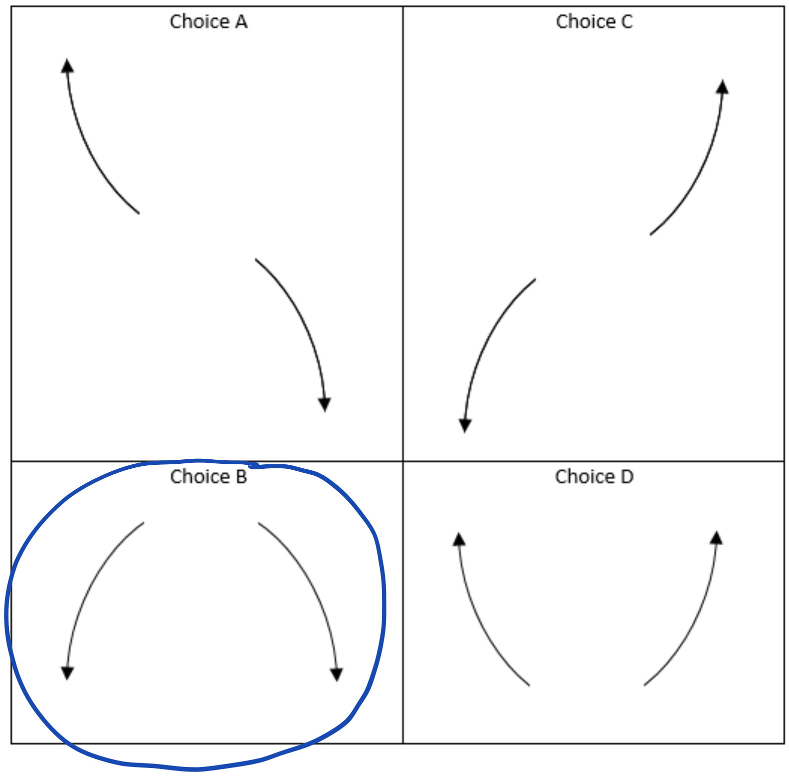


Example 4. The chart below illustrates the end behavior of a polynomial function:

Even Degree	Odd Degree
Positive Leading Coefficient $a_n > 0$  End Behavior: $x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to \infty$	Positive Leading Coefficient $a_n > 0$  End Behavior: $x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to -\infty$
Negative Leading Coefficient $a_n < 0$  End Behavior: $x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to -\infty$	Negative Leading Coefficient $a_n < 0$  End Behavior: $x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to \infty$

Example 5. Choose the end behavior of the polynomial function:

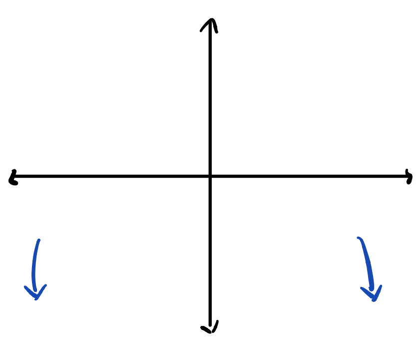
$$f(x) = -(x+7)^6(x+5)^4(x-5)^3(x-7)^3$$



★ NEGATIVE LEADING COEFFICIENT!

DEGREE OF POLYNOMIAL IS THE SUM OF THE MULTIPLICITIES: $6 + 4 + 3 + 3 = 16$

$\Rightarrow$ EVEN DEGREE, NEGATIVE LEADING COEFFICIENT:



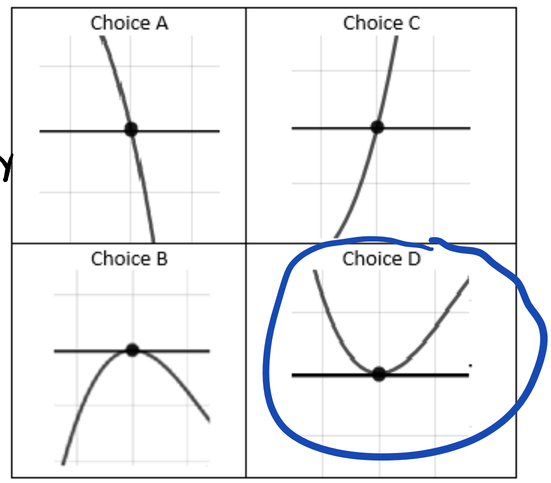
Example 6. Choose the option below that describes the behavior at $x = -3$ of the polynomial:

$x=-6 \quad x=-3 \quad x=3 \quad x=6$

$$f(x) = \underbrace{(x+6)}\underbrace{(x+3)^4}\underbrace{(x-3)^3}\underbrace{(x-6)}$$

ZEROS:

1. $x=-6$, ODD MULTIPLICITY
2. $x=-3$, EVEN MULTIPLICITY
3. $x=3$, ODD MULTIPLICITY
4. $x=6$, ODD MULTIPLICITY

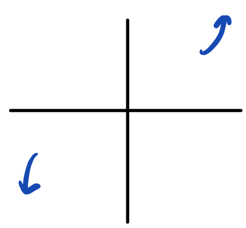


★ POSITIVE LEADING COEFFICIENT!

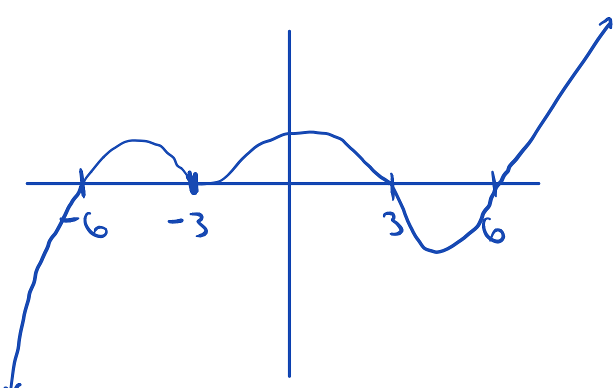
★ DEGREE OF $f(x)$ IS THE SUM OF THE MULTIPLICITIES:

$$1+4+3+1 = 9$$

$\Rightarrow$ ODD DEGREE AND POSITIVE LEADING COEFFICIENT:



PLOT ZEROS AND SKETCH GRAPH:

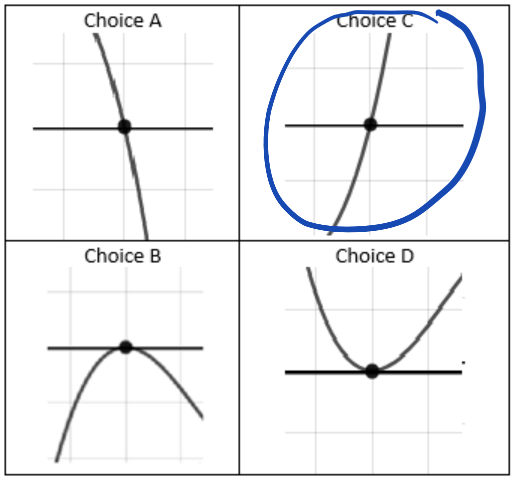


Example 7. Choose the option below that describes the behavior at $x = -9$ of the polynomial:

$$f(x) = \underbrace{\overset{x=-9}{(x+9)^3}}\underbrace{\overset{x=-3}{(x+3)^5}}\underbrace{\overset{x=3}{(x-3)^3}}\underbrace{\overset{x=9}{(x-9)^2}}$$

ZEROS:

1. X = −9, ODD MULTIPLICITY
2. X = −3, ODD MULTIPLICITY
3. X = 3, ODD MULTIPLICITY
4. X = 9, EVEN MULTIPLICITY

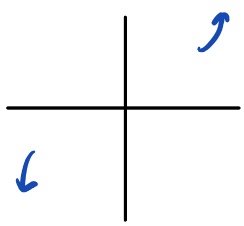


★ POSITIVE LEADING COEFFICIENT !

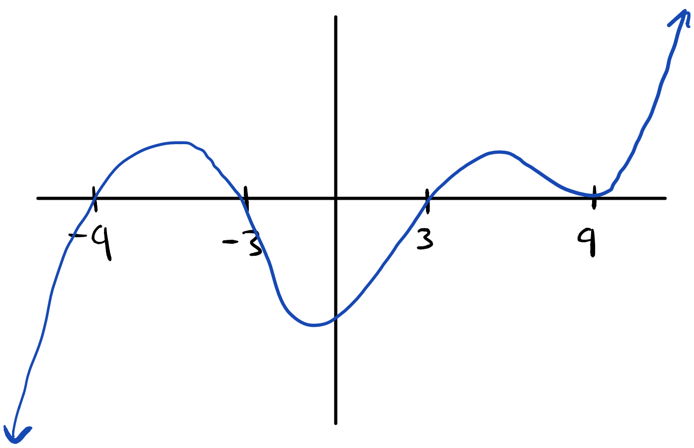
★ DEGREE OF f(X) IS THE SUM OF THE MULTIPLICITIES:

$$3 + 5 + 3 + 2 = 13$$

⇒ ODD DEGREE AND POSITIVE LEADING COEFFICIENT:

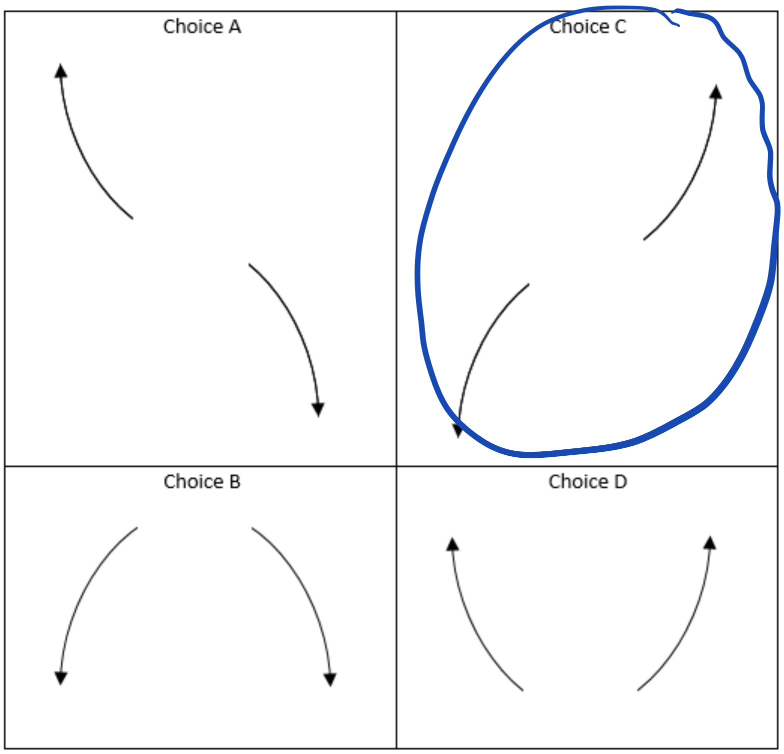


PLOT ZEROS AND SKETCH GRAPH:



Example 8. Choose the end behavior of the polynomial function:

$$f(x) = (x+9)(x+4)^6(x-4)^3(x-9)$$

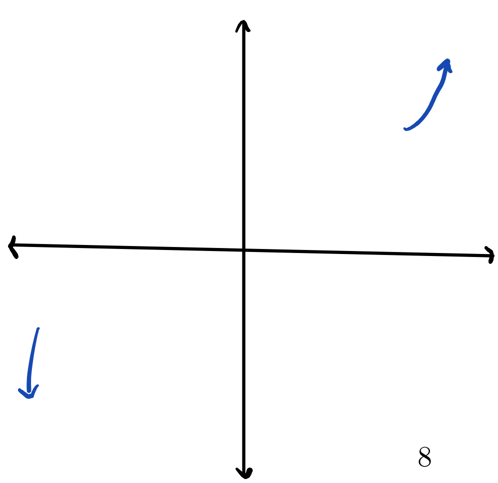


★ POSITIVE LEADING COEFFICIENT!

★ DEGREE OF f(x) IS THE SUM OF THE MULTIPLICITIES:

$1 + 6 + 3 + 1 = 11$

⇒ ODD DEGREE AND POSITIVE LEADING COEFFICIENT:



6.2 Graphing Polynomials

Definition

A TURNING POINT of the graph of a polynomial function is the point where a function changes from rising to falling or from falling to rising. A polynomial of degree n will have at most $n-1$ turning points.

How to Determine the Zeros and Multiplicities of a Polynomial of Degree n Given its Graph

1. If the graph crosses the x-axis at the intercept, it is a zero with ODD MULTIPLICITY.
2. If the graph touches the x-axis and bounces off the axis, it is a zero with EVEN MULTIPLICITY.
3. The sum of the multiplicities is n.

Definition

If a polynomial of lowest degree p has x-intercepts at $x = x_1, x_2, ..., x_n$, then the polynomial can be written in factored form:

$$f(x) = (x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}.$$

$p = p_1 + p_2 + \cdots + p_n$

Note 6. In the factored form of a polynomial, the powers on each factor can be determined by the behavior of the graph at the corresponding ZEROS, and the stretch factor a can be determined given a value of the function other than the X - INTERCEPTS.

How to Determine a Polynomial Function Given its Graph

1. Identify the X - INTERCEPTS to determine the factors of the polynomial.
2. Examine the behavior of the graph at x-intercepts to determine the MULTIPLICITY of each factor.
3. Find the polynomial of least degree containing all the factors found in step 2.
4. Use any other point on the graph (typically the y-intercept) to determine the stretch factor (or, you can analyze the end behavior of the graph to determine the stretch factor).

☆ IF $x=c$ IS A ZERO, THEN $(x-c)$ IS A FACTOR OF THE POLYNOMIAL.

$$f(x) = a(x-c_1)^{m_1}(x-c_2)^{m_2}\ldots(x-c_{m_n})^{m_n}$$

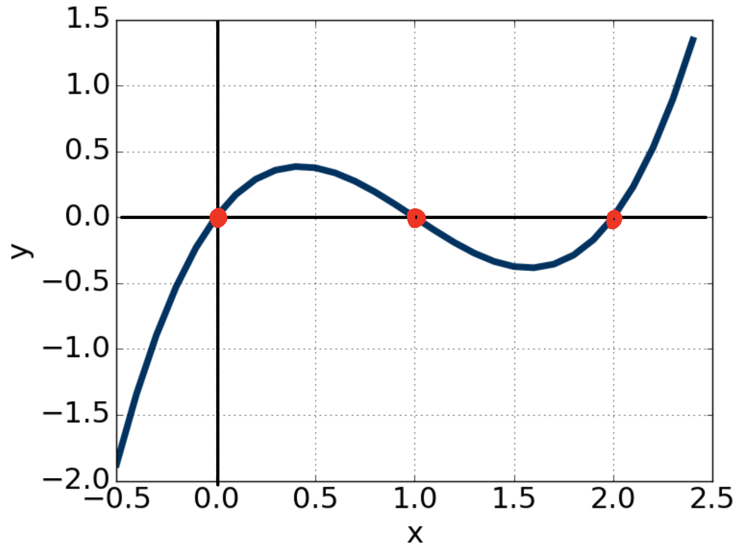
$c_1, c_2, \ldots, c_{m_n}$ ARE THE ZEROS AND X-INTERCEPTS

$m_1, \ldots, m_n$ ARE THE MULTIPLICITIES

$m_1+m_2+\ldots+m_n=m$ IS THE DEGREE OF $f(x)$

a IS THE LEADING COEFFICIENT

Example 9. Write an equation of the function graphed below:



X-INTERCEPTS:

1. X = 0 , CROSSES X-AXIS ⇒ ODD MULTIPLICITY

2. X = 1 , CROSSES X-AXIS ⇒ ODD MULTIPLICITY

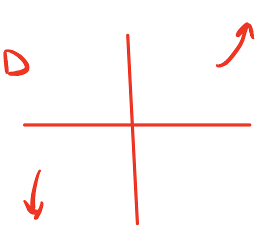
3. X = 2 , CROSSES X-AXIS ⇒ ODD MULTIPLICITY

$$\Rightarrow f(x) = a(x-0)(x-1)(x-2)$$
$$= ax(x-1)(x-2)$$

END BEHAVIOR:

"DEGREE": 3

⇒ ODD DEGREE AND

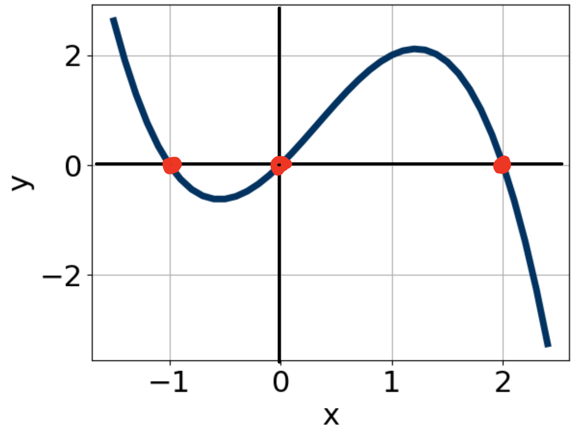


$$\boxed{f(x) = x(x-1)(x-2)}$$

⇒ POSITIVE LEADING COEFFICIENT

⇒ $a = 1$

Example 10. Write an equation of the function graphed below:



X-INTERCEPTS:

1. $X = -1$, CROSSES X-AXIS $\Rightarrow$ ODD MULTIPLICITY

   $(x-(-1)) = (x+1)$ IS A FACTOR WITH MULTIPLICITY 1

2. $X = 0$, CROSSES X-AXIS $\Rightarrow$ ODD MULTIPLICITY

   $(x-0) = x$ IS A FACTOR WITH MULTIPLICITY 1

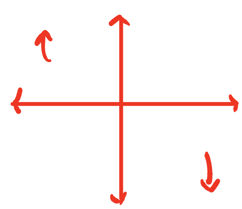
3. $x = 2$, CROSSES X-AXIS $\Rightarrow$ ODD MULTIPLICITY

   $(x-2)$ IS A FACTOR WITH MULTIPLICITY 1

END BEHAVIOR:

"DEGREE": 3

ODD DEGREE AND END BEHAVIOR IS



$\Rightarrow$ NEGATIVE LEADING COEFFICIENT, SO $a < 0 \Rightarrow \underline{a = -1}$

$$f(x) = -(x+1)x(x-2)$$

$$\boxed{f(x) = -x(x+1)(x-2)}$$

How to Sketch the Graph of a Polynomial Function

1. Find the x-intercepts (zeros).

2. Find the y-intercepts.

3. Check for symmetry. If the function is an even function, then its graph is symmetric about the Y - AXIS (that is, $f(-x) = f(x)$). If the function is an odd function, then its graph is symmetric about the X - AXIS (that is, $f(-x) = -f(x)$).

4. Determine the behavior of the polynomial at the zeros using their MULTIPLICITIES.

5. Determine the LEADING COEFFICIENT.

6. Sketch a graph.

7. Check that the number of TURNING POINTS does not exceed one less than the degree of the polynomial.

6.3 Lowest Degree Polynomial

The Factor Theorem

k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$.

Note 7. The following statements are equivalent:

1. X=a IS A ZERO OF THE FUNCTION f
2. X=a IS A SOLUTION OF THE EQUATION f(X)=0
3. (X−a) IS A FACTOR OF THE POLYNOMIAL f(X)
4. (a,0) IS AN X-INTERCEPT OF THE GRAPH OF f(X)

Note 8. If we are given the zeros of a polynomial, we can use the FACTOR THEOREM to construct the lowest-degree polynomial.

Example 11. Construct the lowest-degree polynomial given the zeros below:

$$3, -3, -4$$

FACTORS:

1. $(x-3)$
2. $(x-(-3)) = (x+3)$
3. $(x-(-4)) = (x+4)$

$\Rightarrow \boxed{f(x) = (x-3)(x+3)(x+4)}$

$f(x) = (x^2 + 3x - 3x - 9)(x+4)$

$f(x) = (x^2-9)(x+4)$

FOIL!

$\boxed{f(x) = x^3 + 4x^2 - 9x - 36}$

Example 12. Construct the lowest-degree polynomial given the zeros below:

$$-\frac{4}{3}, -\frac{3}{2}, -3$$

FACTORS:

1. $(x-(-\frac{4}{3})) = x+\frac{4}{3}$ → GET RID OF FRACTIONS: $(x+\frac{4}{3}=0)3 \to 3x+4=0$

$\boxed{3x+4}$

2. $(x-(-\frac{3}{2})) = x+\frac{3}{2}$ → GET RID OF FRACTIONS: $(x+\frac{3}{2}=0)2 \to 2x+3=0$

$\boxed{2x+3}$

3. $(x-(-3)) = \boxed{x+3}$

$$\boxed{f(x) = (x+3)(3x+4)(2x+3)}$$

Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n > 0$, then $f(x)$ has at least one COMPLEX ZERO. In fact, if $f(x)$ is a polynomial of degree $n > 0$ and a is a nonzero real number, then $f(x)$ has exactly n LINEAR FACTORS:

$$f(x) = a(x-c_1)(x-c_2)\cdots(x-c_n)$$

where $c_1, c_2, ..., c_n$ are complex numbers. That is, $f(x)$ has n ROOTS if we allow for multiplicities.

Note 9. This does NOT mean that every polynomial has an imaginary zero. Real numbers are a subset of the complex numbers, but complex numbers are not a subset of the real numbers.

The Linear Factorization Theorem

If f is a polynomial function of degree n, then f has n FACTORS, and each factor is of the form $(x-c)$, where c is a complex number. That is, a polynomial function has the same number of linear factors as its degree.

Complex Conjugate Theorem

Suppose f is a polynomial function with real coefficients. If f has a complex zero of the form $a+bi$, then the COMPLEX CONJUGATE of the zero, $a-bi$, is also a zero.

ALSO APPLIES TO IRRATIONAL ROOTS!

A Closer Look at the Zeros of a Polynomial Function

Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Case 1: b^2-4ac is positive and not a perfect square:

$$x = \frac{\text{INTEGER}}{\text{INTEGER}} \pm \frac{\text{IRRATIONAL}}{\text{INTEGER}}$$

$$x = \text{RATIONAL} \pm \text{IRRATIONAL}$$

Case 2: b^2-4ac is negative:

$$x = \frac{\text{INTEGER}}{\text{INTEGER}} \pm \frac{\text{COMPLEX}}{\text{INTEGER}}$$

$$x = \text{RATIONAL} \pm \text{COMPLEX}$$

Example 13. Construct the lowest-degree polynomial given the zeros below:

$$\sqrt{2}, \frac{1}{3}$$

SINCE $\sqrt{2}$ IS A ZERO, $-\sqrt{2}$ IS ALSO A ZERO!

FACTORS:

1. $(x-\sqrt{2})$
2. $(x-(-\sqrt{2})) = (x+\sqrt{2})$
3. $(x-\frac{1}{3})$ $\Rightarrow$ GET RID OF FRACTION: $(x-\frac{1}{3}=0)3 \longrightarrow 3x-1=0$

$f(x) = (x-\sqrt{2})(x+\sqrt{2})(3x-1)$

$f(x) = (x^2+\sqrt{2}x-\sqrt{2}x-2)(3x-1)$

$f(x) = (x^2-2)(3x-1)$

$\boxed{f(x) = 3x^3-x^2-6x-2}$

Example 14. Construct the lowest-degree polynomial given the zeros below:

$$4+3i, -\frac{2}{5}$$

SINCE $4+3i$ IS A ZERO, $4-3i$ IS ALSO A ZERO

FACTORS:

1. $(x-(4+3i)) = \boxed{(x-4-3i)}$
2. $(x-(4-3i)) = \boxed{(x-4+3i)}$
3. $(x-(-\frac{2}{5})) = (x+\frac{2}{5})$ $\Rightarrow$ GET RID OF FRACTIONS: $(x+\frac{2}{5}=0)5 \rightarrow 5x+2=0$

$\boxed{(5x+2)}$

$f(x) = (x-4-3i)(x-4+3i)(5x+2)$

$f(x) = ((x-4)-3i)((x-4)+3i)(5x+2)$

$f(x) = [(x-4)^2-(3i)^2](5x+2)$

$f(x) = [x^2-8x+16-9i^2](5x+2)$

$f(x) = (x^2-8x+25)(5x+2)$

$f(x) = 5x^3+2x^2-40x^2-16x+125x+50$

$\boxed{f(x) = 5x^3-38x^2+109x+50}$