Module 7 Lecture Notes

MAC1105

Fall 2019

7 Rational Functions

7.1 Domain of Rational Functions

Definition

A VERTICAL ASYMPTOTE is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the input approaches a. We write

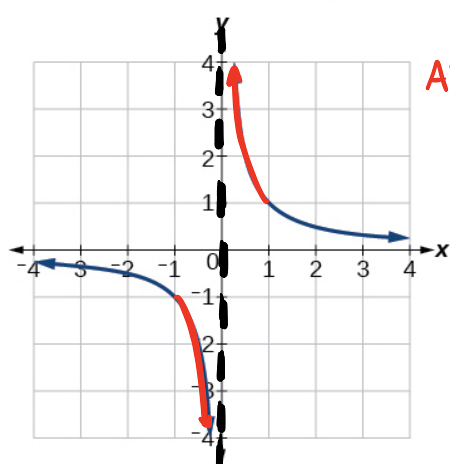
$$\text{As } \ x \to a, \ f(x) \to \infty$$

"AS X APPROACHES a, f(X) APPROACHES INFINITY"

or

$$\text{As } \ x \to a, \ f(x) \to -\infty$$

Example 1. Consider the graph of the function $f(x) = \dfrac{1}{x}$:



AS $x \to 0^+$, $f(x) \to \infty$

AS $x \to 0^-$, $f(x) \to -\infty$

We cannot divide by zero, so as $x \to 0^-$, $f(x) \to -\infty$. That is, as x approaches zero from the left, $f(x)$ approaches $-\infty$. Similarly, as $x \to 0^+$, $f(x) \to \infty$. That is, as x approaches zero from the right, $f(x)$ approaches ∞. So, there is a vertical asymptote at $x = 0$.

Note 1. A vertical asymptote represents a value at which a rational function is UNDEFINED, so the value is not in the domain of the function. In general, to find the domain of a rational function we need to find the values which cause the DENOMINATOR to be equal to zero.

How to Find the Domain of a Rational Function

1. Set the DENOMINATOR equal to zero.
2. Solve to find the x-values that make the DENOMINATOR equal to zero.
3. The domain is ALL REAL NUMBERS except those found in step 2.

Example 2. Find the domain of the rational function:

$$f(x) = \frac{1}{x+3}$$

$x + 3 = 0$
$x = -3$

$\Rightarrow$ DOMAIN IS ALL REAL NUMBERS EXCEPT $x = -3$

$\Rightarrow$ DOMAIN IS $(-\infty, -3) \cup (-3, \infty)$

Note 2. It does not matter what type of real number makes the denominator equal to zero. When we have two or more fractions in the function, the domain is the UNION of the domains of each function separately. In other words, we will remove EVERY value of the function that makes the denominator equal to zero from the domain of the function.

Example 3. Find the domain of the following function:

$$f(x) = \frac{1}{3x+2} - \frac{1}{-3x-5}$$

☆ FIND VALUES OF X THAT MAKE THE DENOMINATOR ZERO:

1. $3x+2=0$

   $3x=-2$

   $\boxed{x=-\frac{2}{3}}$

2. $-3x-5=0$

   $-3x=5$

   $\boxed{x=-\frac{5}{3}}$

DOMAIN IS ALL REAL NUMBERS EXCEPT $-\frac{5}{3}, -\frac{2}{3}$

DOMAIN IS $\boxed{(-\infty, -\frac{5}{3}) \cup (-\frac{5}{3}, -\frac{2}{3}) \cup (-\frac{2}{3}, \infty)}$

Note 3. If we are given a rational function which has a denominator with degree larger than 1, we will first need to FACTOR the denominator and then solve for the x-values that make the denominator equal to ZERO.

Example 4. Find the domain of the rational function:

$$f(x) = \frac{1}{x^2 + 4x + 3}$$

★ DENOMINATOR CANNOT EQUAL ZERO!

$x^2 + 4x + 3 = 0$

$(x+3)(x+1) = 0$

$x+3=0 \qquad x+1=0$

$\boxed{x = -3} \qquad \boxed{x = -1}$

DOMAIN IS ALL REAL NUMBERS EXCEPT -3 AND -1

DOMAIN IS $\boxed{(-\infty, -3) \cup (-3, -1) \cup (-1, \infty)}$

Example 5. Find the domain of the rational function:

$$f(x) = \frac{1}{2x^3 - 3x^2 - 9x}$$

★ FIND VALUES OF X THAT MAKE THE DENOMINATOR ZERO:

$2x^3 - 3x^2 - 9x = 0$

$x(2x^2 - 3x - 9) = 0$

$x(2x^2 - 6x + 3x - 9) = 0$

$x[2x(x-3) + 3(x-3)] = 0$

$x[(x-3)(2x+3)] = 0$

$\underbrace{x}\,\underbrace{(x-3)}\,\underbrace{(2x+3)} = 0$

$x = 0 \qquad x - 3 = 0 \qquad 2x + 3 = 0$

$\boxed{x = 0} \qquad \boxed{x = 3} \qquad \boxed{x = -\frac{3}{2}}$

$2(-9) = -18$

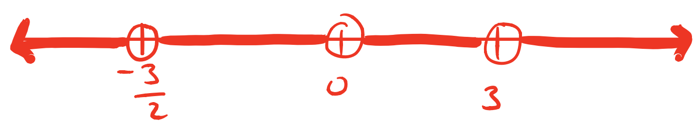
-18	1
18	-1
-6	3
6	-3

$(-6)(3) = -18$

AND

$-6 + 3 = -3$

DOMAIN IS ALL REAL NUMBERS EXCEPT $0, 3, -\frac{3}{2}$



DOMAIN:

$\boxed{(-\infty, -\tfrac{3}{2}) \cup (-\tfrac{3}{2}, 0) \cup (0, 3) \cup (3, \infty)}$

Example 6. Find the domain of the rational function:

$$f(x) = \frac{1}{x^3 + 13x}$$

FIND VALUES THAT MAKE THE DENOMINATOR ZERO:

$x^3 + 13x = 0$

$x(x^2 + 13) = 0$

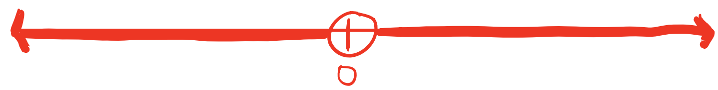
$\boxed{x = 0}$ $\quad x^2 + 13 = 0$

$x^2 = -13$

$x = \pm\sqrt{-13}$

NOT REAL!

DOMAIN IS ALL REAL NUMBERS EXCEPT 0



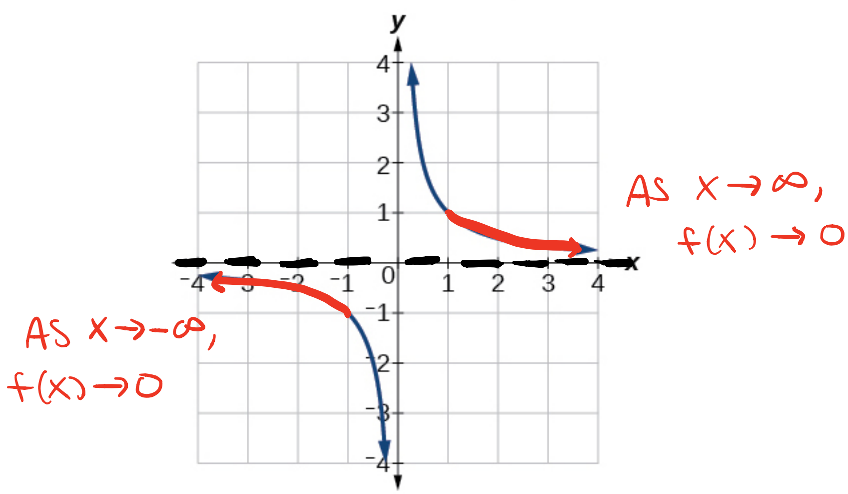
$\boxed{\text{DOMAIN IS } (-\infty, 0) \cup (0, \infty)}$

Definition

A HORIZONTAL ASYMPTOTE is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty, \; f(x) \to b$$

Example 7. Recall the graph of $f(x) = \dfrac{1}{x}$:



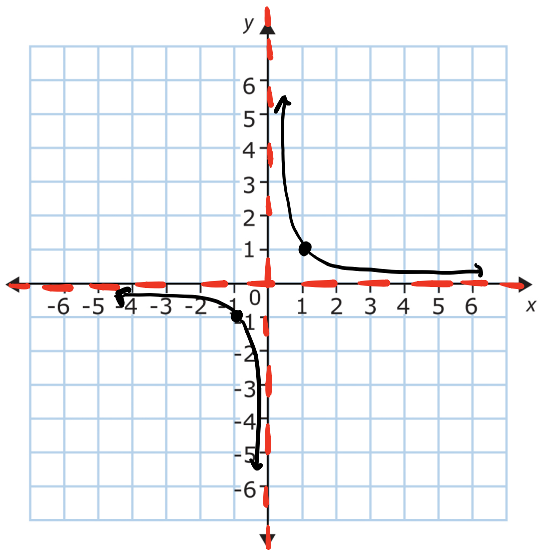
As $x \to \infty$, $f(x) \to 0$ _______________. That is, as x approaches infinity, $f(x)$ approaches zero.

Similarly, as $x \to -\infty$, $f(x) \to 0$ _______________. That is, as x approaches ∞, $f(x)$ approaches zero.

So, there is a horizontal asympotote at $y = 0$ _______________.

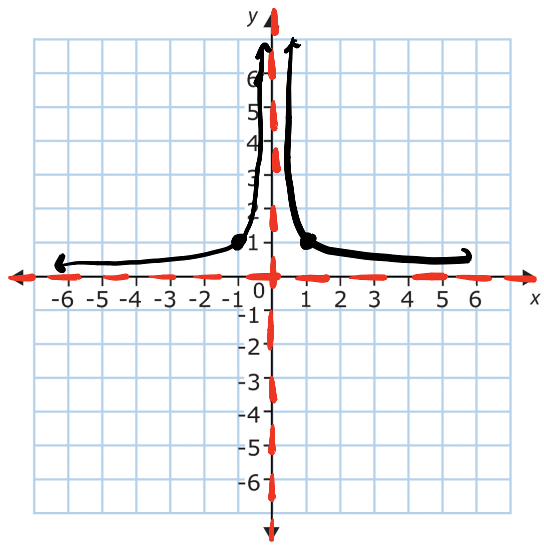
7.2 Graphing Rational Functions

Graphs of Basic Rational Functions To graph rational functions in this section, we will work with two basic rational functions: $f(x) = \dfrac{1}{x}$ and $f(x) = \dfrac{1}{x^2}$.



· VERTICAL ASYMPTOTE AT $X=0$

· HORIZONTAL ASYMPTOTE AT $y=0$



· VERTICAL ASYMPTOTE AT $X=0$

· HORIZONTAL ASYMPTOTE AT $y=0$

Graphing Rational Functions With Translations

Consider the function $f(x) = \dfrac{a}{x-h} + k$.

Horizontal Shifts

☆ (h,k) IS THE POINT WHERE THE ASYMPTOTES CROSS

☆ NEW VERTICAL ASYMPTOTE IS $x = h$

☆ $f(x) = \dfrac{a}{x-h}$ IS THE GRAPH OF $\dfrac{1}{x}$ SHIFTED h UNITS TO THE RIGHT

☆ $f(x) = \dfrac{a}{x+h}$ IS THE GRAPH OF $\dfrac{1}{x}$ SHIFTED h UNITS TO THE LEFT

Vertical Shifts

☆ (h,k) IS THE POINT WHERE THE ASYMPTOTES CROSS

☆ NEW HORIZONTAL ASYMPTOTE IS $y = k$

☆ $f(x) = \dfrac{a}{x} + k$ IS THE GRAPH OF $\dfrac{1}{x}$ SHIFTED k UNITS UP

☆ $f(x) = \dfrac{a}{x} - k$ IS THE GRAPH OF $\dfrac{1}{x}$ SHIFTED k UNITS DOWN

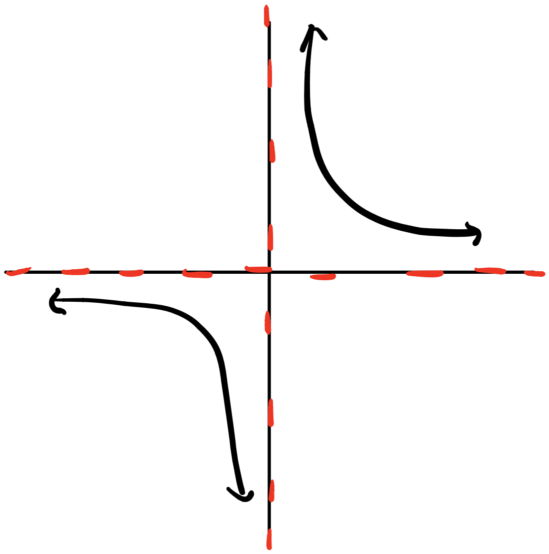
Stretch Factor

☆ a IS THE STRETCH FACTOR. IN THIS CLASS WE WILL ASSUME $a = 1$ OR $a = -1$

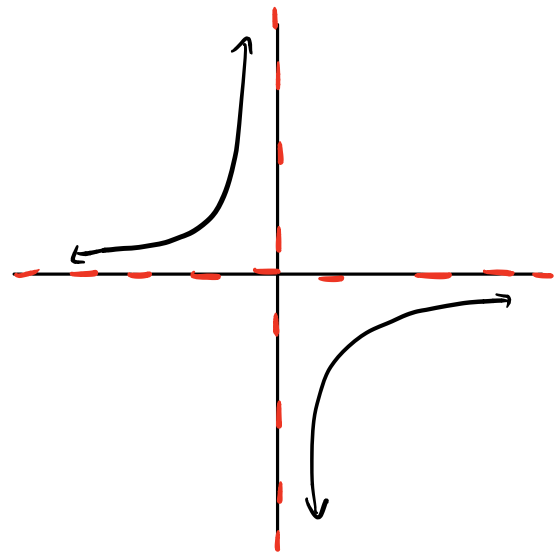
Note 4. So, we can now use the graph of $\frac{1}{x}$ to graph $f(x) = \frac{a}{x-h} + k$. When translating the graph, we must remember to shift horizontal and/or vertical asymptotes.

Note 5. To find the value of a, plug in a point on the graph after finding the value of h and k.

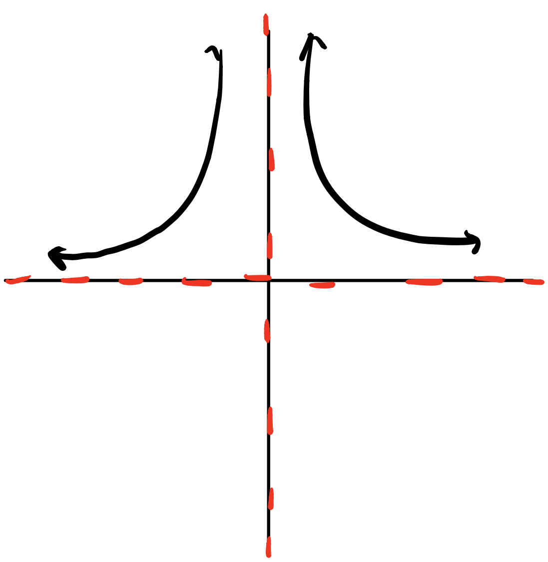
$f(x) = \frac{1}{x} \Rightarrow a = 1$



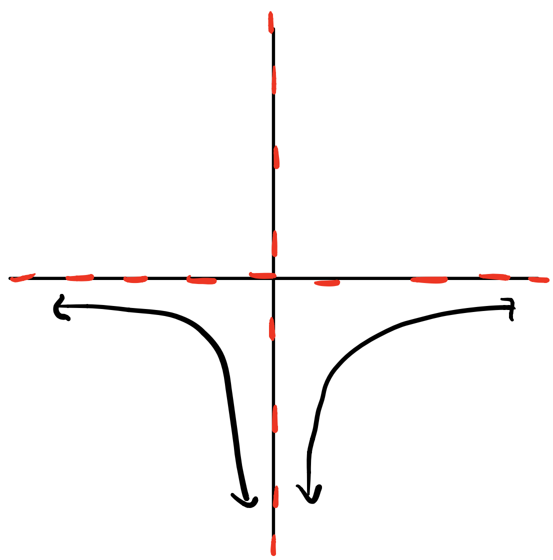
$f(x) = \frac{-1}{x} \Rightarrow a = -1$



$f(x) = \frac{1}{x^2} \Rightarrow a = 1$



$f(x) = \frac{-1}{x^2} \Rightarrow a = -1$



Graphing Rational Functions With Translations

Consider the function $f(x) = \dfrac{a}{(x-h)^2} + k$.

Horizontal Shifts

★ (h, K) IS THE POINT WHERE THE ASYMPTOTES CROSS

★ NEW VERTICAL ASYMPTOTE IS $x = h$

★ $f(x) = \dfrac{a}{(x-h)^2}$ IS THE GRAPH OF $\dfrac{1}{x^2}$ SHIFTED h UNITS TO THE RIGHT

★ $f(x) = \dfrac{a}{(x+h)^2}$ IS THE GRAPH OF $\dfrac{1}{x^2}$ SHIFTED h UNITS TO THE LEFT

Vertical Shifts

★ (h, K) IS THE POINT WHERE THE ASYMPTOTES CROSS

★ NEW HORIZONTAL ASYMPTOTE IS $y = K$

★ $f(x) = \dfrac{a}{x^2} + K$ IS THE GRAPH OF $\dfrac{1}{x^2}$ SHIFTED K UNITS UP

★ $f(x) = \dfrac{a}{x^2} - K$ IS THE GRAPH OF $\dfrac{1}{x^2}$ SHIFTED K UNITS DOWN

Stretch Factor

★ a IS THE STRETCH FACTOR. IN THIS CLASS WE WILL ASSUME $a = 1$ OR $a = -1$

Standard Form of Rational Functions

The standard form of $f(x) = \dfrac{a}{x-h} + k$ is

$$f(x) = \frac{a}{h_1 x - h_0} + K$$

VERTICAL ASYMPTOTE $X = \dfrac{h_0}{h_1}$

The standard form of $f(x) = \dfrac{a}{(x-h)^2} + k$ is

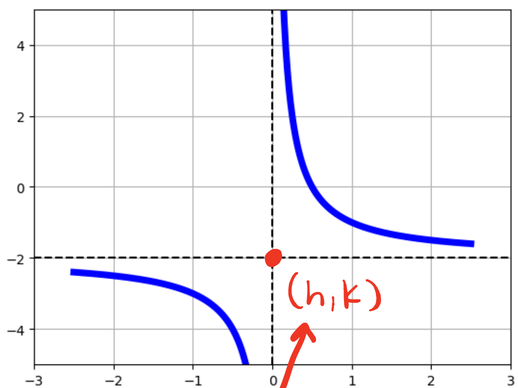
$$f(x) = \frac{a}{(h_1 x - h_0)^2} + K$$

HORIZONTAL ASYMPTOTE $y = K$

Intercepts of Rational Functions

If a rational function does not have a vertical asymptote at $x = 0$ (that is, if our function is defined at $x = 0$), then the y-intercept of the function occurs when $X = 0$. If a rational function does not have a horizontal asymptote at $y = 0$ (that is if our function is defined for all x-values where $f(x) = 0$), then the x-intercept of the function occurs when $y = 0$.

Example 8. Write an equation for the graph of the rational function below. Assume $a = 1$ or $a = -1$:



✓ PARENT FUNCTION IS $f(x) = \frac{1}{x}$

☆ $a = 1$

☆ $(h,k) = (0,-2)$

☆ $f(x) = \dfrac{a}{x-h} + k \Rightarrow f(x) = \dfrac{1}{x-0} - 2$

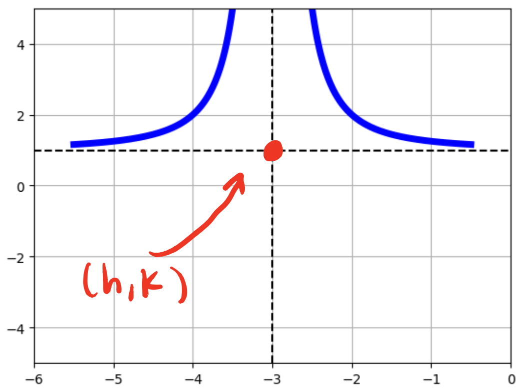
$$\boxed{f(x) = \frac{1}{x} - 2}$$

OUR NEW GRAPH IS THE GRAPH OF $f(x) = \frac{1}{x}$ SHIFTED DOWN 2 UNITS!

HORIZONTAL ASYMPTOTE $y = k \Rightarrow y = -2$

VERTICAL ASYMPTOTE $x = h \Rightarrow x = 0$

Example 9. Write an equation for the graph of the rational function below:



↙ PARENT FUNCTION IS $\frac{1}{x^2}$

☆ $a=1$

☆ $(h,k)=(-3,1)$

☆ $f(x)=\frac{a}{(x-h)^2}+k \Rightarrow f(x)=\frac{1}{(x-(-3))^2}+1$

$$\boxed{f(x)=\frac{1}{(x+3)^2}+1}$$

OUR NEW GRAPH IS THE GRAPH OF $\frac{1}{x^2}$ SHIFTED 3 UNITS TO THE LEFT AND 1 UNIT UP!

HORIZONTAL ASYMPTOTE $y=k \Rightarrow y=1$

VERTICAL ASYMPTOTE $x=h \Rightarrow x=-3$

7.3 Solving Rational Equations

How to Solve Rational Equations

1. Identify the DOMAIN (ie find the values of x for which the DENOMINATOR are equal to 0).

2. Find the least common denominator.

3. Multiply both sides by the LEAST COMMON DENOMINATOR.

4. You should now have a linear or quadratic equation. Simplify your remaining equation and solve.

5. Check your solutions and check that your solutions are in the DOMAIN.

Example 11. Solve the rational equation below:

$$\frac{5}{x} - \frac{1}{3} = \frac{1}{x}$$

★ $X \neq 0$

★ DENOMINATOR IS 0 WHEN $X=0$

$\Rightarrow$ DOMAIN IS ALL REAL NUMBERS EXCEPT $X=0$ $\Rightarrow (-\infty, 0) \cup (0, \infty)$

★ LCD IS $3X$

$$\left(\frac{5}{X} - \frac{1}{3} = \frac{1}{X}\right) 3X$$

$$3\cancel{X}\left(\frac{5}{\cancel{X}}\right) - \cancel{3}X\left(\frac{1}{\cancel{3}}\right) = 3\cancel{X}\left(\frac{1}{\cancel{X}}\right)$$

$$15 - X = 3$$
$$-15 \qquad -15$$
$$-X = -12$$
$$\boxed{X = 12}$$

X=12 IS THE SOLUTION BECAUSE 12 IS IN THE DOMAIN!

Example 12. Solve the rational equation below:

$$-\frac{8}{8x-5}+1=\frac{32}{-32x+20}$$

DOMAIN: $8x-5 \neq 0$

$8x \neq 5$

$x \neq \frac{5}{8}$

$\Rightarrow$ DOMAIN IS ALL REAL NUMBERS EXCEPT $\frac{5}{8}$, OR

$(-\infty, \frac{5}{8}) \cup (\frac{5}{8}, \infty)$

$$-\frac{8}{8x-5}+1=\frac{32}{-4(8x-5)}$$

★ LCD IS $-4(8x-5)$:

$$\left(-\frac{8}{8x-5}+1=\frac{32}{-4(8x-5)}\right)\left(-4(8x-5)\right)$$

$$-4\cancel{(8x-5)}\left(\frac{-8}{\cancel{8x-5}}\right)+[-4(8x-5)](1)=\cancel{-4}\cancel{(8x-5)}\left(\frac{32}{\cancel{-4}\cancel{(8x-5)}}\right)$$

$$(-4)(-8)-4(8x-5)=32$$

$$32-32x+20=32$$

$$52-32x=32$$

$$-52 \qquad = -52$$

$$-32x=-20$$

$$x=\frac{-20}{-32}=\frac{5}{8}$$

$x=\frac{5}{8}$ IS NOT IN THE DOMAIN!

$\Rightarrow$ $\boxed{\text{SOLUTION DOES NOT EXIST}}$

Example 13. Solve the rational equation below:

LCD: $3(x-2)(2x+5)$

DOMAIN:

1. $x-2\neq 0$
   $x\neq 2$
2. $2x+5\neq 0$
   $x\neq -\frac{5}{2}$

★ DOMAIN IS ALL REAL NUMBERS EXCEPT $2, -\frac{5}{2}$

$\Rightarrow (-\infty,-\frac{5}{2})\cup(-\frac{5}{2},2)\cup(2,\infty)$

$$-\frac{2x^2}{3(2x^2+x-10)}-\frac{4x}{3(x-2)}=-\frac{3}{2x+5}$$

$$-\frac{2x^2}{3(x-2)(2x+5)}-\frac{4x}{3(x-2)}=-\frac{3}{2x+5}$$

$$\left[-\frac{2x^2}{3(x-2)(2x+5)}-\frac{4x}{3(x-2)}=\frac{-3}{2x+5}\right]\left(3(x-2)(2x+5)\right)$$

$$-2x^2-4x(2x+5)=-3(3(x-2))$$

$$-2x^2-8x^2-20x=-9(x-2)$$

$$-10x^2-20x=-9x+18$$

$$0=10x^2+11x+18$$

$$x=\frac{-11\pm\sqrt{11^2-4(10)(18)}}{2(10)}$$

$$x=\frac{-11\pm\sqrt{121-720}}{20}$$

$$x=\frac{-11\pm\sqrt{-599}}{20}$$

★ CANNOT TAKE THE SQUARE ROOT OF A NEGATIVE NUMBER

$\Rightarrow$ $\boxed{\text{NO REAL SOLUTIONS}}$

LCD: $x(x-2)$

Example 13. Solve the rational equation below:

DOMAIN:

1. $x \neq 0$
2. $x-2 \neq 0 \Rightarrow x \neq 2$

DOMAIN IS ALL REAL NUMBERS EXCEPT $x=0$ AND $x=2$

$\Rightarrow (-\infty,0) \cup (0,2) \cup (2,\infty)$

$$\frac{10}{x(x-2)} + \frac{4}{x} = \frac{5}{(x-2)}$$

$$\left[\frac{10}{x(x-2)} + \frac{4}{x} = \frac{5}{(x-2)}\right] x(x-2)$$

$$10 + 4(x-2) = 5x$$

$$10 + 4x - 8 = 5x$$

$$2 + 4x = 5x$$

$$2 = x$$

★ 2 IS NOT IN THE DOMAIN!

$\Rightarrow$ NO SOLUTION