# Module 8 Lecture Notes

MAC1105

Fall 2019

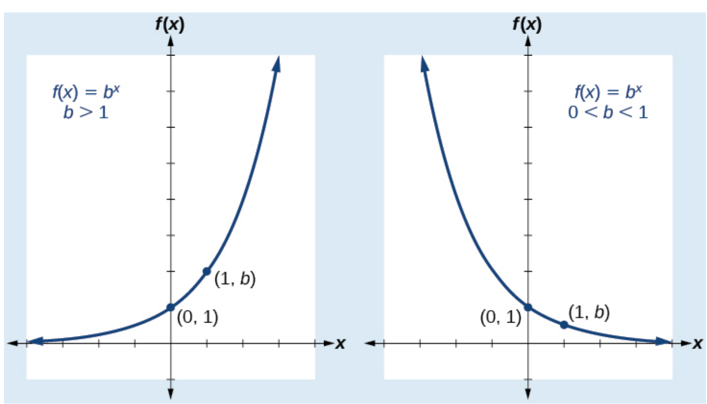
# 8 Logarithmic and Exponential Functions

## 8.1 Domain and Range

**Exponential Functions**

Exponential functions have the form ________________ for any real number $x$ and constant $b > 0$, $b \neq 1$.

**Graph of Exponential Function:** The graph of the parent function, $f(x) = b^x$ is shown below. We call the two cases exponential ________________ and exponential ________________:



**Characteristics of the Graph of $b^x$**

An exponential function of the form $f(x) = b^x$, $b > 0$, $b \neq 1$ has the following characteristics:

- Horizontal Asymptote at ____________________
- Domain: ____________________
- Range: ____________________
- Vertical Asymptote: ____________________
- $x$-intercept: ____________________
- $y$-intercept: ____________________
- Increasing if ____________________
- Decreasing if ____________________

**Shifts of the Parent Function,** $f(x) = b^x$

For any constants $c$ and $d$, the function $b^{x+c} + d$ shifts the graph of the parent function $f(x) = b^x$:

- Vertically ______ units, in the ____________________ direction as the sign of ______
- Horizontally ______ units, in the ____________________ direction as the sign of ______
- The $y$-intercept becomes _______________
- The horizontal asymptote becomes ____________________
- The range becomes ____________________
- The domain is ____________________ (it remains ____________________ )

**Example 1.** Determine the domain AND range of the exponential function:

$$f(x) = -8^{x-4} + 6$$

**Example 2.** Determine the domain AND range of the exponential function:

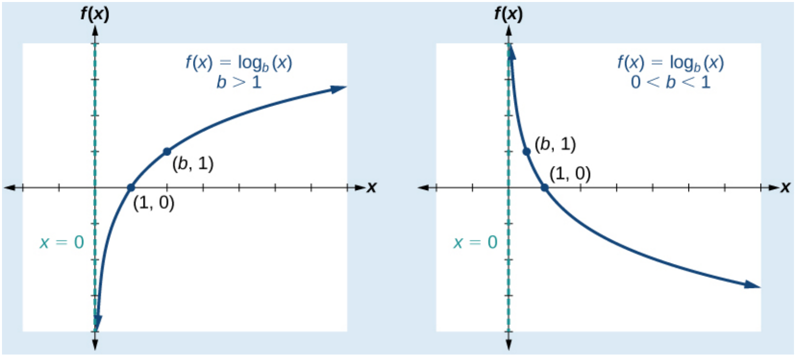
$$f(x) = -8^{x-10} + 4$$

**Logarithmic Functions**

Logarithmic functions have the form ____________________ for any real number $x > 0$ and constant $b > 0$, $b \neq 1$. We read ____________________ as "the logarithm with base $b$ of $x$".

**Note 1.** We have that $y = \log_b x$ is equivalent to $x = b^y$. So, the logarithmic function $y = \log_b x$ is the ____________________ of the exponential function $y = b^x$.

**Graph of Logarithm Function:** The graph of the parent function, $f(x) = \log_b(x)$ is shown below:




**Characteristics of the Graph of** $\log_b(x)$

For any real number $x$ and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$ :

- Vertical Asymptote at ____________________
- Domain: ____________________
- Range: ____________________
- $x$-intercept: ____________________
- Key point: ____________________
- $y$-intercept: ____________________
- Increasing if ____________________
- Decreasing if ____________________

**Note 2.** For any constant, $c$, the function $f(x) = \log_b(x+c)$ shifts the graph of $\log_b(x)$ by ______ units to the __________________ if $c > 0$ and by ______ units to the __________________ if $c < 0$. When we shift the graph of $\log_b(x)$ to the right and left, we must also shift the __________________ of the function.

**How to Determine the Domain of Logarithm Functions**

Recall that the domain of the parent function, $\log_b(x)$ is __________________. Since the graph of $\log_b(x+c)$ shifts the graph of $\log_b(x)$ to the right and left, we must also shift the domain (and vertical asymptote). The function $\log_b(x+c)$ has a vertical asymptote at __________________, so the domain of $\log_b(x+c)$ is __________________.

**Note 3.** Another way to consider finding the domain of $\log_b(x+c)$ is to solve __________________.

**Definition**

A __________________ __________________ is a logarithm with base 10. We write $\log_{10}(x)$ as ______________. The common logarithm of a positive number $x$ satisfies the following definition:

For $x > 0$, ____________________________________

**Note 4.** Since the graph of a logarithmic function $\log_b(x+c) + d$ does note have any ______________ asymptotes, the range is ______________ (it remains _______________________).

**Example 3.** Determine the domain AND range of the logarithmic function:

$$f(x) = \log(x - 5) + 7$$

**Example 4.** Determine the domain AND range of the logarithmic function:

$$f(x) = -\log(x - 9) + 10$$

**Note 5.** The domain of logarithmic functions tells us that we cannot take the logarithm of a

______________________ number. We also cannot take the logarithm of _____.

## 8.2 Convert Between Forms

**Relationship Between Logarithmic Functions and its Corresponding Exponential Form**

We can express the relationship between logarithmic functions and its corresponding exponential form as follows:

**How to Convert From Logarithmic Form to Exponential Form**

1. Examine the equation $y = \log_b(x)$ and identify ______, ______, and ______.
2. Rewrite $y = \log_b(x)$ as __________________.

**Note 6.** To convert from exponential form to logarithmic form, follow the same steps above in reverse.

**Example 5.** Convert the function below from logarithmic form to exponential form:

$$y = \log_7(9)$$

**Example 6.** Convert the function below from logarithmic form to exponential form:

$$y = \log_{10}(x - 6) + 1$$

**Example 7.** Convert the function below from exponential form to logarithmic form:

$$y = 10^{x-4} + 1$$

**Note 7.** Changing between forms is most helpful when trying to solve logarithmic equations.

**Example 8.** Solve the logarithmic equation below:

$$\log_4(4x) = 9$$

**Example 9.** Solve the logarithmic equation below:

$$\log_3(4x-6)+8=-\frac{2}{3}$$

**Example 10.** Solve $y=\log_4(64)$ without using a calculator.

**Note 8.** Recall that $\pi \approx 3.14$. Similarly, we can define a new irrational number, $e \approx 2.718281828...$

**Definition**

The function given by $f(x) = e^x$ is called the ______________________________ ______________________________ ______________________________ with natural base $e$.

**Definition**

A ________________________ ________________________ is a logarithm with base $e$. We write $\log_e(x)$ as

__________. The natural logarithm of a positive number $x$ satisfies the following definition:

For $x > 0$, ________________________________________________.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) =$ __________ for all $x$, and $e^{\ln(x)} =$ __________ for all $x > 0$.

### 8.3 Properties of Logs

**Basic Logarithm Properties**

Two basic properties of logarithms are as follows:

$$\log_b(1) = \underline{\hspace{3cm}}$$

$$\log_b(b) = \underline{\hspace{3cm}}$$

**One-to-One Property**

The one-to-one property for logarithms states that $\log_b M = \log_b N$ if and only if ____________________.

**The Product Rule for Logarithms**

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms:

________________________________________

**Example 11.** Expand $\log_2 xy$

**Example 12.** Expand $\log_3(30x(3x+4))$

**The Quotient Rule for Logarithms**

The quotient rule for logarithms can be used to simplify a logarithm of a quotient by rewriting it as a difference of individual logarithms:

________________________________________

**Example 13.** Expand $\log_2\left(\dfrac{x}{y}\right)$

**Example 14.** Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$

**The Power Rule for Logarithms**

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as a product of the exponent times the logarithm of a base:

________________________________________

**Example 15.** Expand $\log_2 x^5$

**Example 16.** Use properties of logarithms to simplify the expression below:

$$\log\left(\frac{\sqrt{8x^7y^3}}{z^4}\right)$$

**Example 17.** Use properties of logarithms to simplify the expression below:

$$\log\left(\frac{\sqrt{4x^4y^5}}{z^5}\right)$$

**Example 18.** Use properties of logarithm functions to solve the logarithmic equation below:

$$6 = \ln\left(\sqrt{\frac{4}{e^x}}\right)$$

## 8.4 Solve Exponential Functions

**One-to-One Property of Exponential Functions**

For any algebraic expressions $S$ and $T$, and any positive real number $b \neq 1$, $b^S = b^T$ if and only if

______________.

**Using the One-to-One Property to Solve Exponential Equations**

1. Rewrite each side of the equation as a power with a ____________________

   ____________________.

2. Use the rules of exponents to simplify so that the resulting equation has the form

   __________ = __________.

3. Use the One-to-One property to set the exponents equal.

4. Solve the resulting equation, $S = T$ for the unknown.

**Example 19.** Solve the exponential equation below:

$$2^{-5x-6} = 2^{4x+4}$$

**Example 20.** Solve the exponential equation below:

$$4^{5x-3} = 2^{-4x+5}$$

**Example 21.** Solve the exponential equation below:

$$\left(\frac{1}{4}\right)^{4x-2} = 2^{-6x+6}$$

**Note 9.** Using the one-to-one property is very useful, but sometimes we will be given an equation in which the one-to-one property cannot be applied.

**Solving Exponential Equations Using Logarithms**

1. Take the logarithm of both sides of the equation.
2. If one of the terms in the equation has base 10, use the ______________________ ______________________.
3. If neither of the terms in the equation has base 10, then use the ______________________ ______________________.
4. Use the properties of logarithms to solve for the unknowns.

**Example 22.** Solve the exponential equation below:

$$6^{-5x-6} = 5^{6x+4}$$

**Example 23.** Solve the exponential equation below:

$$27^{-6x-3} = \left(\frac{1}{16}\right)^{2x+3}$$