Module 8 Lecture Notes

MAC1105

Fall 2019

8 Logarithmic and Exponential Functions

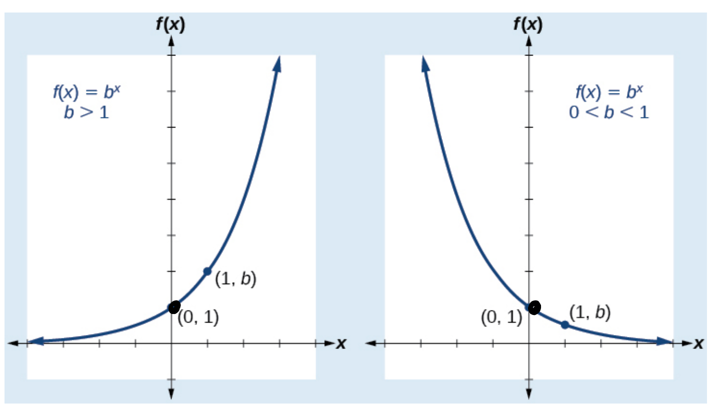
8.1 Domain and Range

Exponential Functions

Exponential functions have the form b^x for any real number x and constant $b > 0$, $b \neq 1$.

Graph of Exponential Function: The graph of the parent function, $f(x) = b^x$ is shown below.

We call the two cases exponential GROWTH and exponential DECAY:



X-INTERCEPT IS (0,1)

EXPONENTIAL GROWTH

EXPONENTIAL DECAY

Characteristics of the Graph of b^x

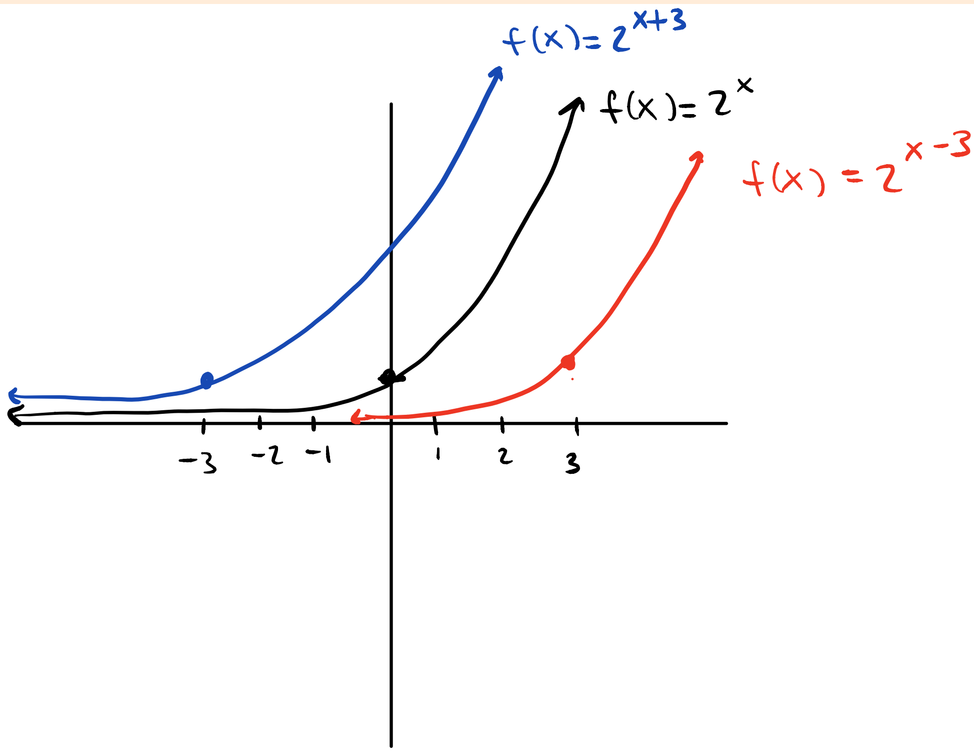
An exponential function of the form $f(x) = b^x$, $b > 0$, $b \neq 1$ has the following characteristics:

- Horizontal Asymptote at $y=0$
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Vertical Asymptote: NONE
- x-intercept: NONE
- y-intercept: $(0, 1)$
- Increasing if $b > 1$
- Decreasing if $b < 1$

Shifts of the Parent Function, $f(x) = b^x$

For any constants c and d, the function $b^{x+c} + d$ shifts the graph of the parent function $f(x) = b^x$:

- Vertically $\underline{d}$ units, in the $\underline{\text{SAME}}$ direction as the sign of $\underline{d}$
- Horizontally $\underline{c}$ units, in the $\underline{\text{OPPOSITE}}$ direction as the sign of $\underline{c}$
- The y-intercept becomes $\underline{(0, b^c + d)}$
- The horizontal asymptote becomes $\underline{y = d}$
- The range becomes $\underline{(d, \infty)}$ IF $b > 0$ AND $(-\infty, d)$ IF $b < 0$
- The domain is $\underline{(-\infty, \infty)}$ (it remains $\underline{\text{UNCHANGED}}$)



Example 1. Determine the domain AND range of the exponential function:

$$f(x) = -8^{x-4} + 6$$

DOMAIN REMAINS UNCHANGED $\Rightarrow$ $\boxed{\text{DOMAIN IS } (-\infty, \infty)}$

RANGE OF $f(x) = b^{x+c} + d$ IS $(-\infty, d)$ IF $b < 0$

$\Rightarrow$ $\boxed{\text{RANGE IS } (-\infty, 6)}$

↑BECAUSE THE GRAPH IS SHIFTED UP 6 UNITS

Example 2. Determine the domain AND range of the exponential function:

$$f(x) = -8^{x-10} + 4$$

DOMAIN REMAINS UNCHANGED $\Rightarrow$ $\boxed{\text{DOMAIN IS } (-\infty, \infty)}$

RANGE OF $f(x) = b^{x+c} + d$ IS (d, ∞)

$\Rightarrow$ $\boxed{\text{RANGE IS } (4, \infty)}$

↑BECAUSE THE GRAPH IS SHIFTED UP 4 UNITS

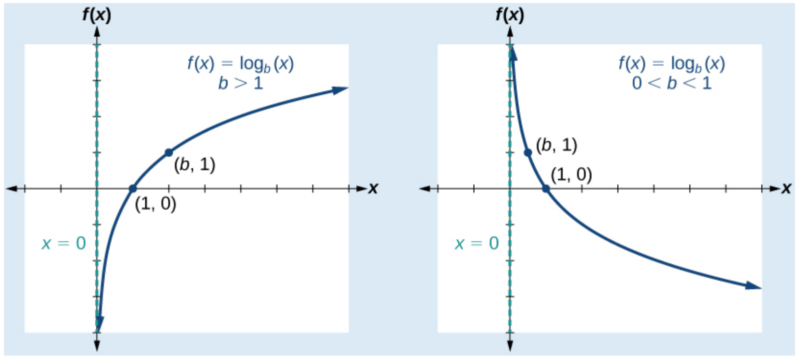
Logarithmic Functions

Logarithmic functions have the form $y = \log_b(x)$ for any real number $x > 0$ and constant $b > 0$, $b \neq 1$. We read $\log_b(x)$ as "the logarithm with base b of x".

Note 1. We have that $y = \log_b x$ is equivalent to $x = b^y$. So, the logarithmic function $y = \log_b x$ is the INVERSE of the exponential function $y = b^x$.

Graph of Logarithm Function: The graph of the parent function, $f(x) = \log_b(x)$ is shown below:

★ X-INTERCEPT IS (1,0)



Characteristics of the Graph of $\log_b(x)$

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- Vertical Asymptote at $x=0$
- Domain: $(0,\infty)$ ★ CANNOT TAKE THE LOGARITHM OF 0 OR A NEGATIVE!
- Range: $(-\infty,\infty)$
- x-intercept: $(1,0)$
- Key point: $(b,1)$
- y-intercept: NONE
- Increasing if $b>1$
- Decreasing if $0<b<1$

Note 2. For any constant, c, the function $f(x) = \log_b(x + c)$ shifts the graph of $\log_b(x)$ by C units to the LEFT if $c > 0$ and by C units to the RIGHT if $c < 0$. When we shift the graph of $\log_b(x)$ to the right and left, we must also shift the DOMAIN of the function.
(& VERTICAL ASYMPTOTE)

How to Determine the Domain of Logarithm Functions

Recall that the domain of the parent function, $\log_b(x)$ is $(0, \infty)$. Since the graph of $\log_b(x + c)$ shifts the graph of $\log_b(x)$ to the right and left, we must also shift the domain (and vertical asymptote). The function $\log_b(x + c)$ has a vertical asymptote at $x = -c$, so the domain of $\log_b(x + c)$ is $(-c, \infty)$.

Note 3. Another way to consider finding the domain of $\log_b(x + c)$ is to solve $x + c > 0$.

Definition

A COMMON LOGARITHM is a logarithm with base 10. We write $\log_{10}(x)$ as $\log(x)$. The common logarithm of a positive number x satisfies the following definition:

For $x > 0$, $y = \log(x)$ IS EQUIVALENT TO $10^y = x$

Note 4. Since the graph of a logarithmic function $\log_b(x + c) + d$ does note have any HORIZONTAL asymptotes, the range is $(-\infty, \infty)$ (it remains UNCHANGED).

Example 3. Determine the domain AND range of the logarithmic function:

$$f(x) = \log(x-5) + 7$$

★ RANGE OF LOGARITHM FUNCTIONS IS UNCHANGED: $\boxed{\text{RANGE IS } (-\infty, \infty)}$

★ DOMAIN: $x - 5 > 0$

$x > 5 \Rightarrow \boxed{\text{DOMAIN IS } (5, \infty)}$

Example 4. Determine the domain AND range of the logarithmic function:

$$f(x) = -\log(x-9) + 10$$

★ RANGE IS $(-\infty, \infty)$

★ DOMAIN: $x - 9 > 0$

$x > 9 \Rightarrow \boxed{\text{DOMAIN IS } (9, \infty)}$

Note 5. The domain of logarithmic functions tells us that we cannot take the logarithm of a NEGATIVE number. We also cannot take the logarithm of 0.

8.2 Convert Between Forms

Relationship Between Logarithmic Functions and its Corresponding Exponential Form

We can express the relationship between logarithmic functions and its corresponding exponential form as follows:

$$\log_b(x) \overset{=}{=} y \quad \Leftrightarrow \quad b^y = x$$

TO

"b TO THE y EQUALS x"

How to Convert From Logarithmic Form to Exponential Form

1. Examine the equation $y = \log_b(x)$ and identify b, y, and x.
2. Rewrite $y = \log_b(x)$ as $b^y = x$.

Note 6. To convert from exponential form to logarithmic form, follow the same steps above in reverse.

Example 5. Convert the function below from logarithmic form to exponential form:

$$y = \log_7(9)$$

=

TO

"7 TO THE y EQUALS 9"

$$\boxed{7^y = 9}$$

Example 6. Convert the function below from logarithmic form to exponential form:

$$y = \log_{10}(x-6) + 1$$

$$-1 \quad = \quad -1$$

$$y-1 = \log_{10}(x-6)$$

$$y-1 = \log_{10}(x-6)$$

"10 TO THE y−1 EQUALS x−6"

$$10^{y-1} = x-6$$

$$+6 \qquad +6$$

$$\boxed{10^{y-1} + 6 = x}$$

Example 7. Convert the function below from exponential form to logarithmic form:

$$y = 10^{x-4} + 1$$

$$-1 \quad = \quad -1$$

$$y-1 = 10^{x-4}$$

"10 TO THE x−4 EQUALS y−1"

$$\log_{10}(y-1) = x-4$$

$$\boxed{\log_{10}(y-1) + 4 = x}$$

Note 7. Changing between forms is most helpful when trying to solve logarithmic equations.

Example 8. Solve the logarithmic equation below:

$$\log_4(4x) = 9$$

TO

"4 TO THE 9 EQUALS 4x"

$$4^9 = 4x$$

$$\frac{262{,}144}{4} = \frac{4x}{4}$$

$$\boxed{65{,}536 = x}$$

Example 9. Solve the logarithmic equation below:

$$\log_3(4x-6)+8=-\frac{2}{3}$$

$$-8 = -8$$

$$\log_3(4x-6) = -\frac{2}{3} - 8$$

$$\log_3(4x-6) = -\frac{2}{3} - \frac{24}{3}$$

$$\log_3(4x-6) = -\frac{26}{3}$$

TO

=

"3 TO THE $-\frac{26}{3}$ EQUALS $4x-6$"

$$3^{-\frac{26}{3}} = 4x-6$$

$$\frac{1}{3^{26/3}} = 4x-6$$

$$0.000073 = 4x-6$$

$$6.000073 = 4x$$

$$\boxed{1.5 = x}$$

Example 10. Solve $y = \log_4(64)$ without using a calculator.

=

TO

"4 TO THE y EQUALS 64"

$$4^y = 64$$

$$\boxed{y = 3}$$

Note 8. Recall that $\pi \approx 3.14$. Similarly, we can define a new irrational number, $e \approx 2.718281828...$

Definition

The function given by $f(x) = e^x$ is called the NATURAL EXPONENTIAL FUNCTION with natural base e.

Definition

A NATURAL LOGARITHM is a logarithm with base e. We write $\log_e(x)$ as ln(x). The natural logarithm of a positive number x satisfies the following definition:

For $x > 0$, y=ln(x) IS EQUIVALENT TO $e^{y} = x$.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) =$ X for all x, and $e^{\ln(x)} =$ X for all $x > 0$.

8.3 Properties of Logs

Basic Logarithm Properties

Two basic properties of logarithms are as follows:

$$\log_b(1) = \underline{\quad 0 \quad}$$

$$\log_b(b) = \underline{\quad 1 \quad}$$

One-to-One Property

The one-to-one property for logarithms states that $\log_b M = \log_b N$ if and only if $\underline{M = N}$. (FOR $b > 0$)

The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms:

$$\underline{\log_b(M \cdot N) = \log_b(M) + \log_b(N)}$$

(FOR $b > 0$)

Example 11. Expand $\log_2 xy$

PRODUCT RULE!

$$\log_2(xy) = \log_2(x) + \log_2(y)$$

Example 12. Expand $\log_3(30x(3x+4))$

$$= \underbrace{\log_3(30x)} + \log_3(3x+4)$$

$$= \log_3(30) + \log_3(x) + \log_3(3x+4)$$

The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm of a quotient by rewriting it as a difference of individual logarithms:

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

(FOR $b > 0$)

Example 13. Expand $\log_2\left(\frac{x}{y}\right)$

☆QUOTIENT RULE!

$$\log_2\left(\frac{x}{y}\right) = \log_2(x) - \log_2(y)$$

Example 14. Expand $\underbrace{\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)}_{\text{QUOTIENT RULE}}$

$$= \log_3(\underbrace{7x^2+21x}_{\downarrow\text{ FACTOR}}) - \underbrace{\log_3(7x(x-1)(x-2))}_{\downarrow\text{ PRODUCT RULE}}$$

$$= \underbrace{\log_3(7x(x+3))}_{\downarrow\text{ PRODUCT RULE}} - \left[\log_3(7x) + \log_3(x-1) + \log_3(x-2)\right]$$

DISTRIBUTE NEGATIVE

$$= \cancel{\log_3(7x)} + \log_3(x+3) - \cancel{\log_3(7x)} - \log_3(x-1) - \log_3(x-2)$$

$$= \boxed{\log_3(x+3) - \log_3(x-1) - \log_3(x-2)}$$

The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as a product of the exponent times the logarithm of a base:

$$\log_b(M^n) = n\log_b(M)$$

(FOR $b>0$)

Example 15. Expand $\log_2 x^5$

☆ POWER RULE !

$$\log_2 x^5 = 5\log_2(x)$$

Example 16. Use properties of logarithms to simplify the expression below:

$$\log\left(\frac{\sqrt{8x^7y^3}}{z^4}\right)$$

QUOTIENT RULE

$$= \log\left(\underbrace{\sqrt{8x^7y^3}}_{\sqrt{8x^7y^3} = (8x^7y^3)^{1/2}}\right) - \underbrace{\log(z^4)}_{\text{POWER RULE}}$$

$$= \log\left((8x^7y^3)^{1/2}\right) - 4\log(z)$$

POWER RULE

$$= \frac{1}{2}\left[\log(8x^7y^3)\right] - 4\log(z)$$

PRODUCT RULE

$$= \frac{1}{2}\left[\log(8) + \underbrace{\log(x^7)} + \underbrace{\log(y^3)}\right] - 4\log(z)$$

POWER RULE

$$= \frac{1}{2}\left[\log(8) + 7\log(x) + 3\log(y)\right] - 4\log(z)$$

DISTRIBUTE

$$= \boxed{\frac{\log(8)}{2} + \frac{7\log(x)}{2} + \frac{3\log(y)}{2} - 4\log(z)}$$

Example 17. Use properties of logarithms to simplify the expression below:

$$\log\left(\frac{\sqrt{4x^4y^5}}{z^5}\right)$$

QUOTIENT RULE

$$= \log\left(\underbrace{\sqrt{4x^4y^5}}\right) - \underbrace{\log(z^5)}$$

$$\sqrt{4x^4y^5} = (4x^4y^5)^{1/2}$$

POWER RULE

$$= \underbrace{\log\left((4x^4y^5)^{1/2}\right)}_{\text{POWER RULE}} - 5\log(z)$$

$$= \underbrace{\frac{1}{2}\left(\log(4x^4y^5)\right)}_{\text{PRODUCT RULE}} - 5\log(z)$$

$$= \frac{1}{2}\left(\log(4) + \underbrace{\log(x^4)} + \underbrace{\log(y^5)}\right) - 5\log(z)$$

POWER RULE

$$= \frac{1}{2}\left(\log(4) + 4\log(x) + 5\log(y)\right) - 5\log(z)$$

DISTRIBUTE

$$= \frac{\log(4)}{2} + \frac{\overset{2}{\cancel{4}}\log(x)}{\cancel{2}} + \frac{5\log(y)}{2} - 5\log(z)$$

$$= \boxed{\frac{\log(4)}{2} + 2\log(x) + \frac{5\log(y)}{2} - 5\log(z)}$$

Example 18. Use properties of logarithm functions to solve the logarithmic equation below:

$$6 = \ln\left(\sqrt{\frac{4}{e^x}}\right)$$

$$6 = \ln\left(\left(\frac{4}{e^x}\right)^{1/2}\right)$$

$$6 = \frac{1}{2}\left(\ln\left(\frac{4}{e^x}\right)\right)$$

$$6 = \frac{1}{2}\left(\ln(4) - \ln(e^x)\right)$$

$$6 = \frac{1}{2}\ln(4) - \frac{1}{2}\ln(e^x)$$

$\star \ln(e^x) = x$

$$\left[6 = \frac{\ln(4)}{2} - \frac{x}{2}\right](2)$$

$\star$ LCD TO GET RID OF FRACTIONS

$$12 = \ln(4) - x$$

$$+x = +x$$

$$x + 12 = \ln(4)$$

$$-12 = -12$$

$$\boxed{x = \ln(4) - 12}$$

8.4 Solve Exponential Functions

One-to-One Property of Exponential Functions

For any algebraic expressions S and T, and any positive real number $b \neq 1$, $b^S = b^T$ if and only if

S=T.

Using the One-to-One Property to Solve Exponential Equations

1. Rewrite each side of the equation as a power with a COMMON BASE.

2. Use the rules of exponents to simplify so that the resulting equation has the form

$$b^S = b^T.$$

3. Use the One-to-One property to set the exponents equal.

4. Solve the resulting equation, $S = T$ for the unknown.

Example 19. Solve the exponential equation below:

$$2^{-5x-6} = 2^{4x+4}$$

★ SAME BASE!
⇒ USE ONE-TO-ONE PROPERTY

$$-5x - 6 = 4x + 4$$
$$+5x \quad = +5x$$
$$-6 = 9x + 4$$
$$-4 \quad = \quad -4$$
$$-10 = 9x \longrightarrow \boxed{\frac{-10}{9} = x}$$

Example 20. Solve the exponential equation below:

$$4^{5x-3} = 2^{-4x+5}$$

CAN CREATE SAME BASE BECAUSE $4 = 2^2$

$$(2^2)^{5x-3} = 2^{-4x+5}$$

$$2^{2(5x-3)} = 2^{-4x+5}$$

USE ONE-TO-ONE PROPERTY!

$$2(5x-3) = -4x+5$$

$$\begin{aligned} 10x - 6 &= -4x + 5 \\ +4x &= +4x \\ \hline 14x - 6 &= 5 \end{aligned}$$

$$\begin{aligned} 14x - 6 &= 5 \\ +6 &= +6 \\ \hline 14x &= 11 \end{aligned}$$

$$\Rightarrow \boxed{x = \frac{11}{14}}$$

Example 21. Solve the exponential equation below:

$$\left(\frac{1}{4}\right)^{4x-2} = 2^{-6x+6}$$

$\frac{1}{4} = \frac{1}{2^2} = 2^{-2}$

$$\left(2^{-2}\right)^{4x-2} = 2^{-6x+6}$$

$$2^{-2(4x-2)} = 2^{-6x+6}$$

USE ONE-TO-ONE PROPERTY!

$$-2(4x-2) = -6x+6$$

$$\begin{aligned} -8x + 4 &= -6x + 6 \\ +8x &= +8x \\ \hline 4 &= 2x + 6 \\ -6 &= -6 \\ \hline \frac{-2}{2} &= \frac{2x}{2} \end{aligned}$$

$$\boxed{-1 = x}$$

Note 9. Using the one-to-one property is very useful, but sometimes we will be given an equation in which the one-to-one property cannot be applied.

Solving Exponential Equations Using Logarithms

1. Take the logarithm of both sides of the equation.
2. If one of the terms in the equation has base 10, use the COMMON LOGARITHM.
3. If neither of the terms in the equation has base 10, then use the NATURAL LOGARITHM.
4. Use the properties of logarithms to solve for the unknowns.

Example 22. Solve the exponential equation below:

★ CANNOT HAVE SAME BASE
⇒ TAKE ln OF BOTH SIDES

$$6^{-5x-6} = 5^{6x+4}$$

$$\ln(6^{-5x-6}) = \ln(5^{6x+4}) \xrightarrow{\text{POWER RULE}} (-5x-6)\ln(6) = (6x+4)\ln(5)$$

$$-5x\ln(6) - 6\ln(6) = 6x\ln(5) + 4\ln(5)$$

$$-6x\ln(5) = -6x\ln(5)$$

GET ALL TERMS WITH X ON ONE SIDE TO SOLVE FOR X

$$-5x\ln(6) - 6x\ln(5) - 6\ln(6) = 4\ln(5)$$

$$+6\ln(6) = +6\ln(6)$$

$$-5x\ln(6) - 6x\ln(5) = 6\ln(6) + 4\ln(5)$$

$$x(-5\ln(6) - 6\ln(5)) = 6\ln(6) + 4\ln(5)$$

$$\boxed{x = \frac{6\ln(6) + 4\ln(5)}{-5\ln(6) - 6\ln(5)}}$$

Example 23. Solve the exponential equation below:

$$27^{-6x-3} = \left(\frac{1}{16}\right)^{2x+3}$$

★ CANNOT HAVE SAME BASE ⇒ TAKE ln OF BOTH SIDES

$$27^{-6x-3} = \frac{1^{2x+3}}{16^{2x+3}}$$

$$\ln\left(27^{-6x-3}\right) = \ln\left(\frac{1^{2x+3}}{16^{2x+3}}\right)$$

$$(-6x-3)\ln(27) = \ln\left(1^{2x+3}\right) - \ln\left(16^{2x+3}\right)$$

$$-6x\ln(27) - 3\ln(27) = (2x+3)\ln(1) - (2x+3)\ln(16)$$

★ $\ln(1) = 0$

$$-6x\ln(27) - 3\ln(27) = 0 - \left(2x\ln(16) + 3\ln(16)\right)$$

$$-6x\ln(27) - 3\ln(27) = -2x\ln(16) - 3\ln(16)$$

$$+2x\ln(16) = +2x\ln(16)$$

$$-6x\ln(27) + 2x\ln(16) - 3\ln(27) = -3\ln(16)$$

$$+3\ln(27) = +3\ln(27)$$

$$-6x\ln(27) + 2x\ln(16) = -3\ln(16) + 3\ln(27)$$

$$x\left(-6\ln(27) + 2\ln(16)\right) = -3\ln(16) + 3\ln(27)$$

$$\boxed{x = \frac{-3\ln(16) + 3\ln(27)}{-6\ln(27) + 2\ln(16)}}$$