Module 9L Lecture Notes

MAC1105

Summer B 2019

9 Operations on Functions

9.1 Domain

Operations on Functions

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by:

- $(f+g)(x) =$ $f(x) + g(x)$
- $(f-g)(x) =$ $f(x) - g(x)$
- $(fg)(x) =$ $f(x)g(x)$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$

Example 1. Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x) = x - 1$ and $g(x) = x^2 - 1$.

1. $(g-f)(x) = g(x) - f(x) = (x^2-1) - (x-1) = x^2 - 1 - x + 1 = x^2 - x$

2. $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{x^2-1}{x-1} = \frac{(\cancel{x-1})(x+1)}{(\cancel{x-1})} = x+1$, $x \neq 1$

$x \neq 1$

Domain of Algebra of Functions

Let f and g be two functions with domains A and B. Then,

Name	Definition	Domain
$f \pm g$	$f(x) \pm g(x)$	$A \cap B$
fg	$f(x)\,g(x)$	$A \cap B$
$\frac{f}{g}$	$\frac{f(x)}{g(x)}$	$A \cap B \cap \{x : g(x) \neq 0\}$

Example 2. Determine the domain of each function below and then determine the domain of $f+g$, fg, and $\dfrac{f}{g}$:

$$f(x) = -5x^2 - 6x + 3$$

$$g(x) = \sqrt{3x-4}$$

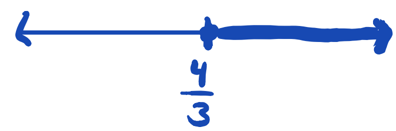
DOMAIN OF $f(x)$: $(-\infty, \infty)$

"A"

DOMAIN OF $g(x)$: $3x - 4 \geq 0$

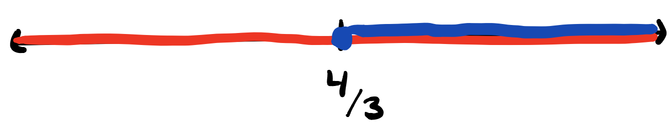
$$3x \geq 4$$

$$x \geq \frac{4}{3} \Rightarrow \text{DOMAIN OF } g(x) \text{ IS } [4/3, \infty) \quad \text{"B"}$$



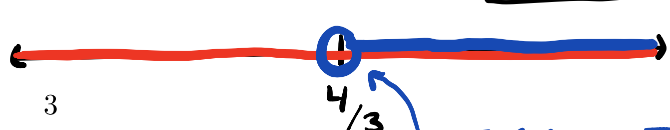
1. DOMAIN OF $f(x) + g(x)$ AND $f(x)g(x)$ IS $A \cap B$:

$$(-\infty, \infty) \cap [4/3, \infty) = \boxed{[4/3, \infty)}$$



2. DOMAIN OF $\dfrac{f(x)}{g(x)}$ IS $A \cap B \cap \{x : g(x) \neq 0\}$:

✯ $g(x) = 0$ WHEN $x = 4/3$, SO $A \cap B \cap \{x : g(x) \neq 0\} = \boxed{(4/3, \infty)}$



WE CANNOT HAVE $x = 4/3$

Example 3. Determine the domain of each function below and then determine the domain of $f+g$, fg, and $\dfrac{f}{g}$:

$$f(x) = -5x^3 + 5x^2 - 5x - 6$$

$$g(x) = \sqrt{4x-6}$$

DOMAIN OF $f(x)$: $(-\infty, \infty)$ "A"

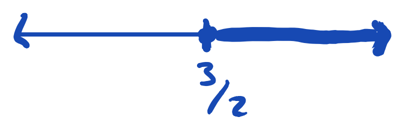
DOMAIN OF $g(x)$: $4x - 6 \geq 0$

$4x \geq 6$

$x \geq 6/4$

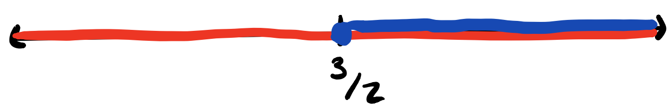
$x \geq 3/2$

$\Rightarrow$ DOMAIN OF $g(x)$ IS $[3/2, \infty)$ "B"



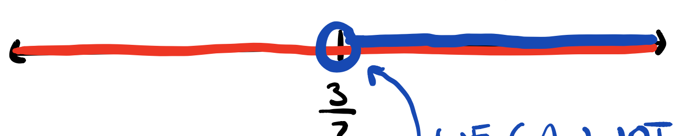
1. DOMAIN OF $f(x) + g(x)$ AND $f(x)g(x)$ IS $A \cap B$:

$(-\infty, \infty) \cap [3/2, \infty) = \boxed{[3/2, \infty)}$



2. DOMAIN OF $\dfrac{f(x)}{g(x)}$ IS $A \cap B \cap \{x : g(x) \neq 0\}$:

✱ $g(x) = 0$ WHEN $x = 3/2$, SO $A \cap B \cap \{x : g(x) \neq 0\} = \boxed{(3/2, \infty)}$



WE CANNOT HAVE $x = \dfrac{3}{2}$

Example 4. Determine the domain of each function below and then determine the domain of $f+g$, fg, and $\dfrac{f}{g}$:

$$f(x) = 6x^3 + 6x^2 - 3x + 6$$

$$g(x) = -\frac{1}{4x+3}$$

DOMAIN OF $f(x)$: $(-\infty, \infty)$

"A"

DOMAIN OF $g(x)$: ALL REAL NUMBERS EXCEPT $4x+3=0$:

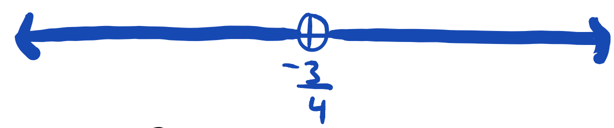
$4x+3=0$

$4x=-3$

$x=-\frac{3}{4}$

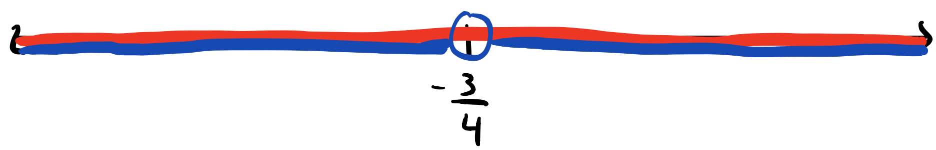
$\Rightarrow$ DOMAIN OF $g(x)$ IS ALL REAL NUMBERS EXCEPT $x=-\frac{3}{4}$

$\Rightarrow (-\infty, -\frac{3}{4}) \cup (-\frac{3}{4}, \infty)$ "B"



1. DOMAIN OF $f(x)+g(x)$ AND $f(x)g(x)$ IS $A \cap B$:

$(-\infty, \infty) \cap \left[(-\infty, -\frac{3}{4}) \cup (-\frac{3}{4}, \infty)\right] = \boxed{(-\infty, -\frac{3}{4}) \cup (-\frac{3}{4}, \infty)}$



2. DOMAIN OF $\dfrac{f(x)}{g(x)}$ IS $A \cap B \cap \{x : g(x) \neq 0\}$:

✩ $g(x)$ DOES NOT EVER EQUAL 0 HERE! SO, $A \cap B \cap \{x : g(x) \neq 0\} = A \cap B$

$\Rightarrow$ DOMAIN IS SAME AS 1:

$$\boxed{(-\infty, -\frac{3}{4}) \cup (-\frac{3}{4}, \infty)}$$

9.2 Composition

Definition

The process of combining functions so that the output of one function becomes the input of another is known as COMPOSITION OF FUNCTIONS. For any input x and functions f and g, this action defines a COMPOSITE FUNCTION, which we write as $f \circ g$ such that

g(x), THE OUTPUT OF g, IS THE INPUT OF f

$$(f \circ g)(x) = f(\overline{g(\underline{x})})$$

x IS THE INPUT OF g

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Note 1. It is important to keep in mind the order of operations when composing functions. That is, $(f \circ g)(x) = f(g(x))$ means that the function f takes $g(x)$ as an input and yields an output of $f(g(x))$.

Example 5. For the two functions below, evaluate $(f \circ g)(-3)$:

$$f(x) = 4x^2 - 4x + 4$$

$$g(x) = -3x^2 - 3x + 4$$

$(f \circ g)(-3) = f(\underline{g(-3)})$

1. FIND $g(-3)$: $g(x) = -3x^2 - 3x + 4$

$$g(-3) = -3(-3)^2 - 3(-3) + 4$$

$$= -3(9) + 9 + 4$$

$$= -27 + 9 + 4$$

$$= -14$$

$\Rightarrow g(-3) = -14$

2. FIND $f(g(-3))$:

$f(g(-3)) = \underline{f(-14)}$

$$f(x) = 4x^2 - 4x + 4$$

$$f(-14) = 4(-14)^2 - 4(-14) + 4$$

$$= 4(196) + 56 + 4$$

$$= 784 + 56 + 4$$

$$= 844$$

$\Rightarrow (f \circ g)(-3) = f(g(-3)) = f(-14) = \boxed{844}$

Example 6. For the two functions below, evaluate $(f \circ g)(3)$ and $(g \circ f)(3)$:

$$f(x) = 5x^2 - 5x - 3$$

$$g(x) = \sqrt{5x - 4}$$

1. $(f \circ g)(3) = f(g(3))$

$\star\ g(3) = \sqrt{5(3)-4}$

$= \sqrt{15-4} = \sqrt{11}$

$\Rightarrow (f \circ g)(3) = f(g(3)) = f(\sqrt{11})$

$\star\ f(\sqrt{11}) = 5(\sqrt{11})^2 - 5(\sqrt{11}) - 3$

$= 5(11) - 5\sqrt{11} - 3$

$= 55 - 5\sqrt{11} - 3$

$= 52 - 5\sqrt{11}$

$\Rightarrow \boxed{(f \circ g)(3) = 52 - 5\sqrt{11}}$

2. $(g \circ f)(3) = g(f(3))$

$\star\ f(3) = 5(3)^2 - 5(3) - 3$

$= 5(9) - 15 - 3$

$= 45 - 15 - 3$

$= 27$

$\Rightarrow (g \circ f)(3) = g(f(3)) = g(27)$

$\star\ g(27) = \sqrt{5(27)-4}$

$= \sqrt{135-4}$

$= \sqrt{131}$

$\Rightarrow \boxed{(g \circ f)(3) = \sqrt{131}}$

Note 2. The example above shows that function compositions is not COMMUTATIVE.

That is, $(f \circ g)(x) \neq$ $(g \circ f)(x)$.

Note 3. The product of functions fg is not the same as the function composition $f(g(x))$ because

$f(g(x))$ $\neq$ $f(x)g(x)$.

9.3 One-to-One

Definition

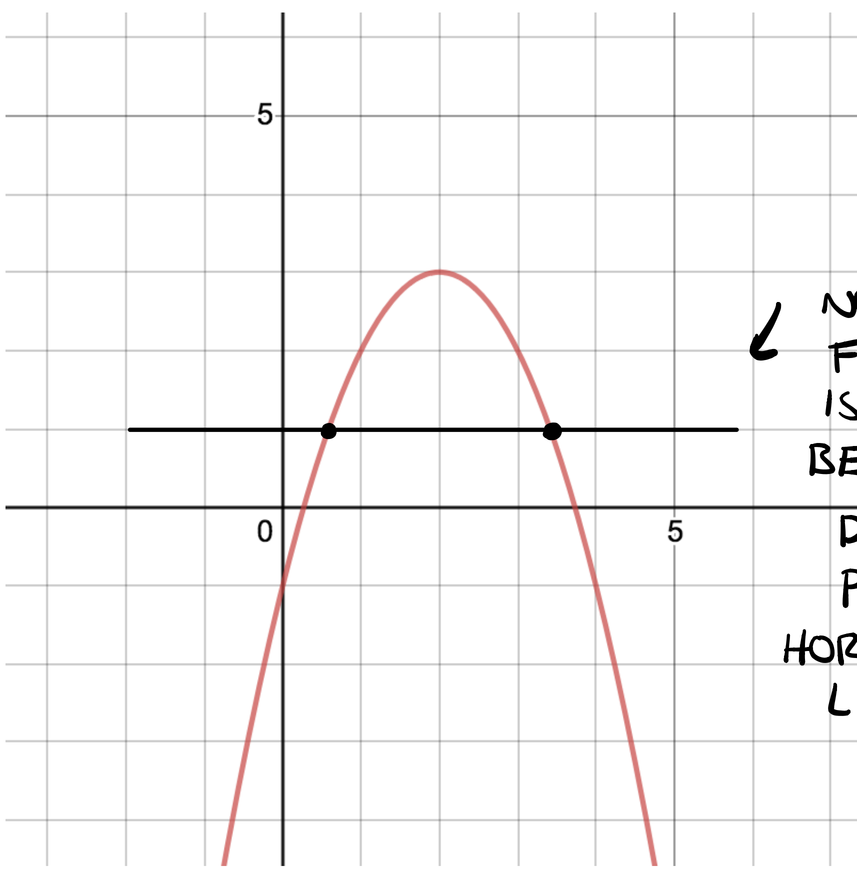
A function f is a ONE - TO - ONE function if each value of the dependent variable (y) corresponds to exactly one value of the independent variable (x).

Note 4. If a function f is a set of ordered pairs, then f is one-to-one of no two ordered pairs have the same second element. That is, if each y has only one x.

Horizontal Line Test

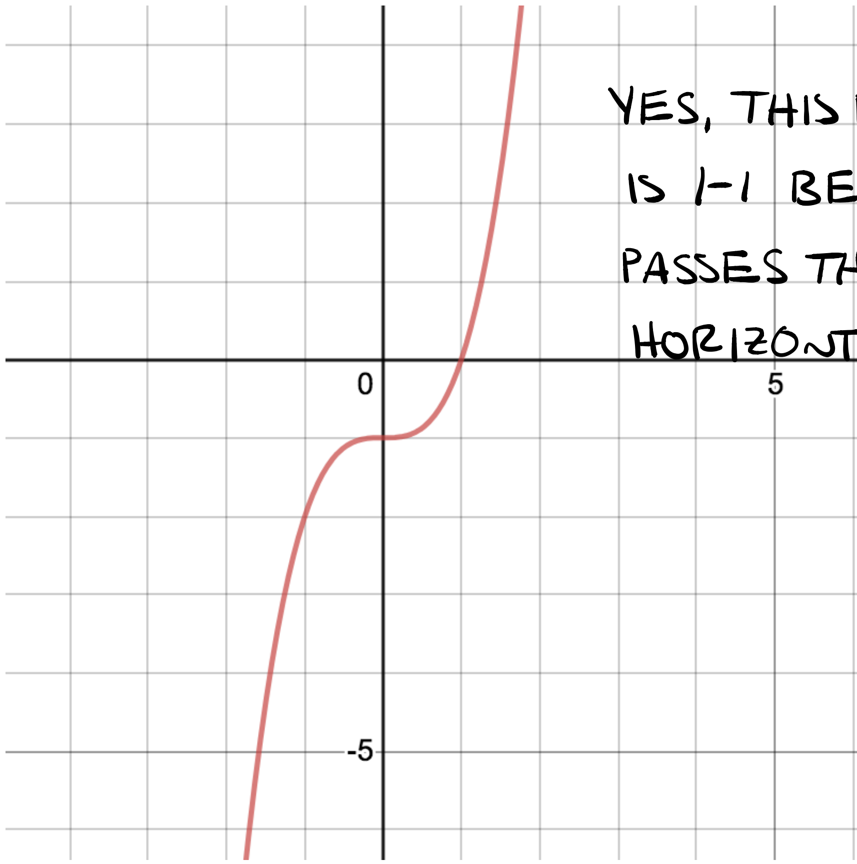
A function f is one-to-one if and only if any horizontal line intersects the graph of f at most once.

Example 7. Is the following graph one-to-one?



NO, THIS FUNCTION IS NOT 1-1 BECAUSE IT DOES NOT PASS THE HORIZONTAL LINE TEST!

Example 8. Is the following graph one-to-one?



YES, THIS FUNCTION IS 1-1 BECAUSE IT PASSES THE HORIZONTAL LINE TEST!

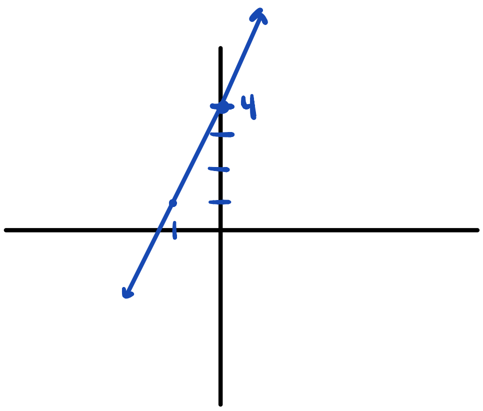
Algebraically Determine if a Function is One-to-One

To show that a function is one-to-one, you can show that $f(y) = f(x)$ if and only if $y = x$.

Example 9. Is the following function one-to-one?

$$f(x) = 3x + 4$$

GRAPH:



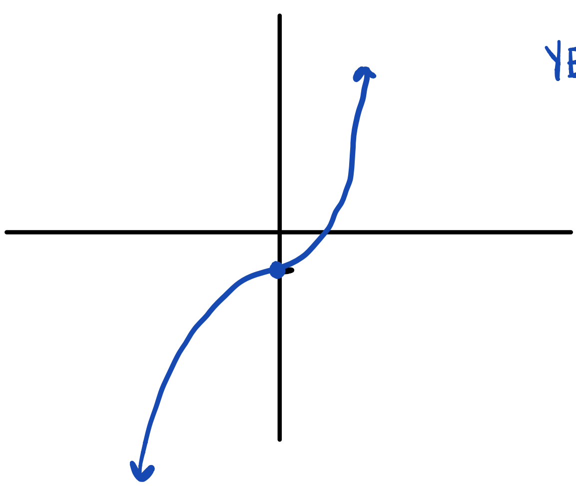
$\Rightarrow$ YES, 1-1

(IT PASSES HORIZONTAL LINE TEST)

Example 10. Is the following function one-to-one?

$$f(x) = x^3 - 1$$

GRAPH SKETCH:



YES, 1-1

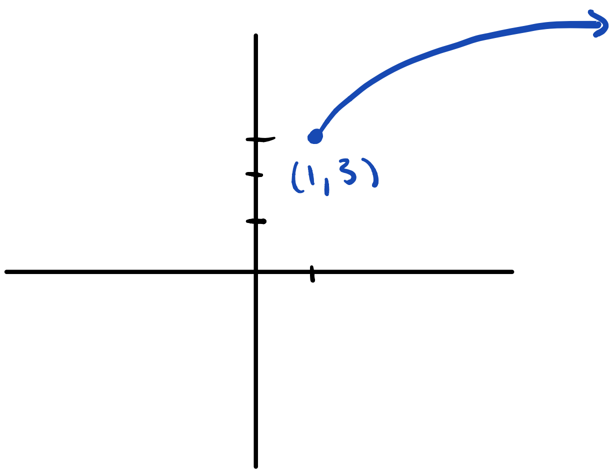
PASSES HORIZONTAL LINE TEST!

Example 11. Is the following function one-to-one?

$$f(x) = \sqrt{x-1} + 3$$

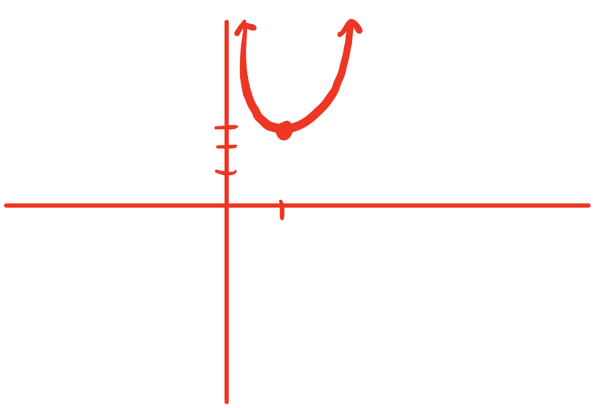
$(h, k) = (1, 3)$

GRAPH:



YES! 1-1, PASSES HORIZONTAL LINE TEST

* $f(x) = (x-1)^2 + 3$ IS NOT 1-1 BECAUSE IT DOES NOT PASS THE HORIZONTAL LINE TEST:



9.4 Inverse

Definition

For any one-to-one function $f(x) = y$, a function $f^{-1}(x)$ is an INVERSE FUNCTION of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

Note 5. Not every function has an inverse, and $f^{-1}(x) \neq \frac{1}{f(x)}$. Given a one-to-one function, f, the inverse of the coordinate pair $(x, f(x))$ is $(f(x), x)$.

Example 12. For a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

$f(2)=4 \Rightarrow f^{-1}(4)=2$ $(2,4) \rightarrow (4,2)$

$f(5)=12 \Rightarrow f^{-1}(12)=5$ $(5,12) \rightarrow (12,5)$

How to Determine if Two Functions $f(x)$ and $g(x)$ are Inverses of Each Other

1. Determine whether $f(g(x))=x$ or $g(f(x))=x$.
2. If either statement is true, then both are true, and $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

Domain and Range of Inverse Functions

The RANGE of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$. The DOMAIN of $f(x)$ is the range of $f^{-1}(x)$.

How to Find the Domain and Range of an Inverse Function

1. If the original function is one-to-one, write the range of the original function as the DOMAIN of the inverse function.

2. If the original function is one-to-one, write the domain of the original function as the RANGE of the inverse function.

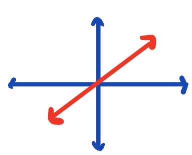
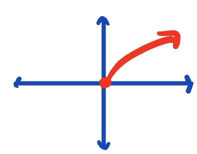
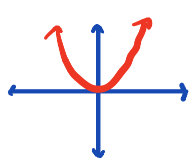
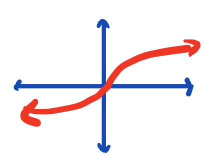
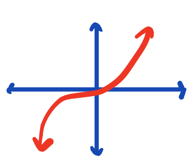
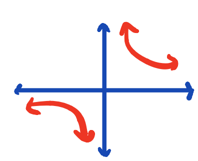
3. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the RANGE of the inverse function.

How to Determine the Inverse of a Function

1. Check that f is a ONE - TO - ONE function.

2. Solve for x.

3. Interchange x and y.

$f(x) = x$  1-1	$f(x) = \sqrt{x}$  1-1
$f(x) = x^2$  NOT 1-1	$f(x) = \sqrt[3]{x}$  1-1
$f(x) = x^3$  1-1	$f(x) = \frac{1}{x}$  1-1

Example 13. Determine if the following function is one-to-one. If the function is one-to-one, find the inverse and define the domain on which the inverse is valid:

$$f(x) = (4x-5)^3 - 4$$

1. IS f 1-1?

YES! IN GENERAL, x^3 IS 1-1

2. $y = (4x-5)^3 - 4$ INTERCHANGE X AND Y

$$x = (4y-5)^3 - 4$$

$$x - 4 = (4y-5)^3$$

$$(x-4)^{1/3} = \left[(4y-5)^3\right]^{1/3}$$

$$\sqrt[3]{x-4} = 4y - 5$$

$$\sqrt[3]{x-4} + 5 = 4y$$

$$\frac{\sqrt[3]{x-4} + 5}{4} = y \Rightarrow \boxed{f^{-1}(x) = \frac{1}{4}\sqrt[3]{x-4} + \frac{5}{4}}$$

★ RANGE OF $f(x) = (4x-5)^3 - 4$ IS $(-\infty, \infty)$

$\Rightarrow$ DOMAIN OF $f^{-1}(x)$ IS $(-\infty, \infty)$

Example 14. Determine if the following function is one-to-one. If the function is one-to-one, find the inverse and define the domain on which the inverse is valid:

$$f(x) = \sqrt{-3x-5} + 7$$

★ RANGE IS $[7, \infty)$ OR $y \geq 7$

1. IS $f(x)$ 1-1 ?

YES! SQUARE ROOT FUNCTION $\Rightarrow$ 1-1

2. $y = \sqrt{-3x-5} + 7$

INTERCHANGE X AND Y

$x = \sqrt{-3y-5} + 7$

$x - 7 = \sqrt{-3y-5}$

$(x-7)^2 = \left(\sqrt{-3y-5}\right)^2$

$x^2 - 14x + 49 = -3y - 5$

$x^2 - 14x + 54 = -3y$

$\dfrac{x^2 - 14x + 54}{-3} = y$

$\Rightarrow \boxed{f^{-1}(x) = -\frac{1}{3}x^2 + \frac{14}{3}x - 18}$

★ DOMAIN OF $f^{-1}(x)$ IS THE RANGE OF $f(x)$:

RANGE OF $f(x)$ IS $[7, \infty)$

$\Rightarrow$ DOMAIN OF $f^{-1}(x)$ IS $[7, \infty)$

Example 15. Determine if the following function is one-to-one. If the function is one-to-one, find the inverse and define the domain on which the inverse is valid:

$$f(x) = (2x + 7)^2 + 2$$

1. IS f 1-1?

NO! PARABOLAS ARE NOT 1-1

$\Rightarrow$ NO INVERSE FUNCTION EXISTS!