Lecture 25: Section 4.1
Angles and Their Measure

Angle - initial side, terminal side, vertex

Standard position of an angle

Positive and negative angles

Coterminal angle

Central angle

Radians

Complementary and suppplementary angles

Degree measure and radian measure

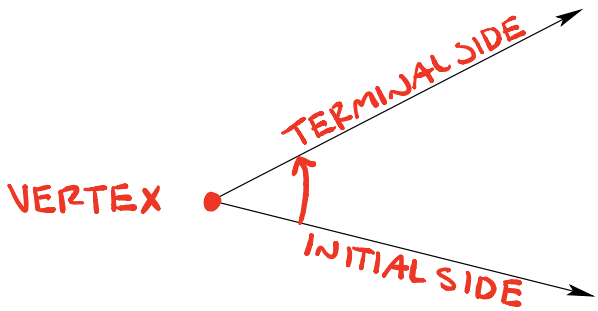
Arc length, s

Area of a sector

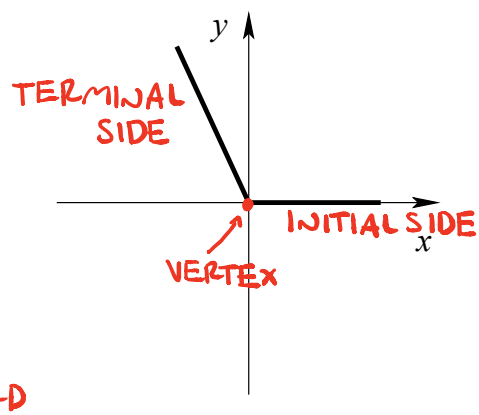
Linear speed

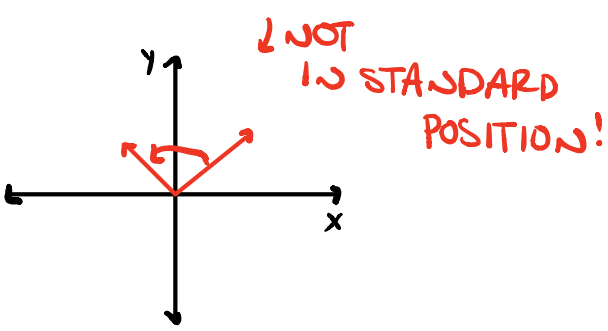
Angular speed

An **angle** is formed by rotating a ray around its endpoint. The starting position of the ray is the **initial side** of the angle, and the position after rotation is the **terminal side** of the angle. The endpoint of the ray is called the **vertex**.

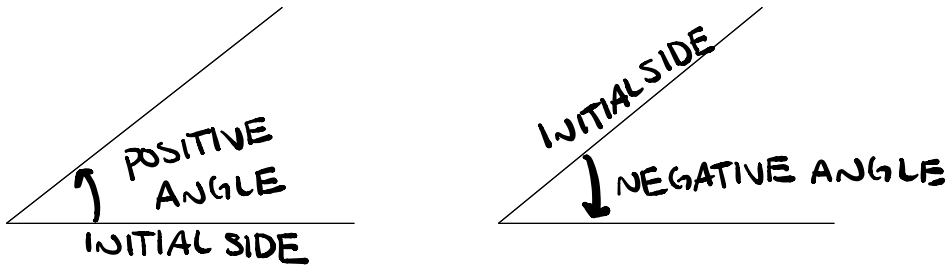


An angle θ is said to be in **standard position** if its vertex is in the origin and its initial side coincides with the positive x-axis.

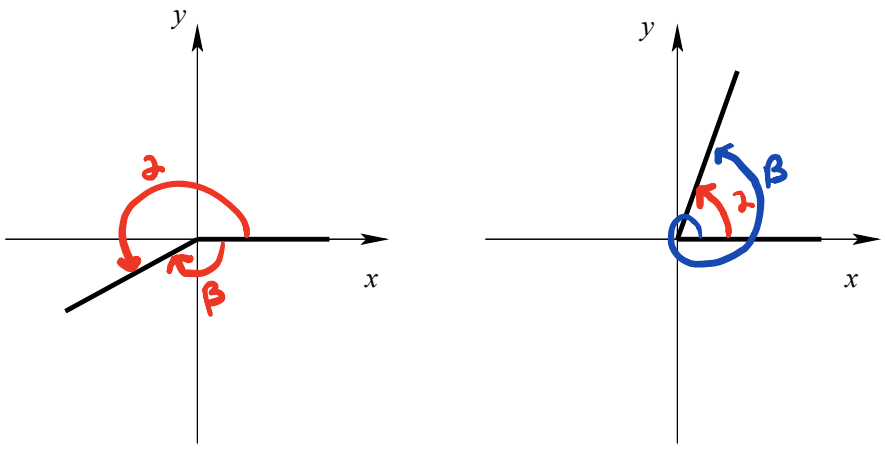




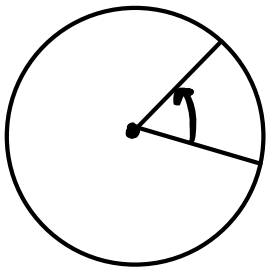
If the rotation is in the counterclockwise direction, the angle is **positive**; if the rotation is clockwise, the angle is **negative**.



Angles α and β are **coterminal angles** if they have the same initial and terminal sides.



A **central angle** is an angle whose vertex is at the center of a circle.



Radian Measure

Def. **One radian** is a measure of the central angle that intercepts an arc whose length is equal to the radius. Algebraically, this means that

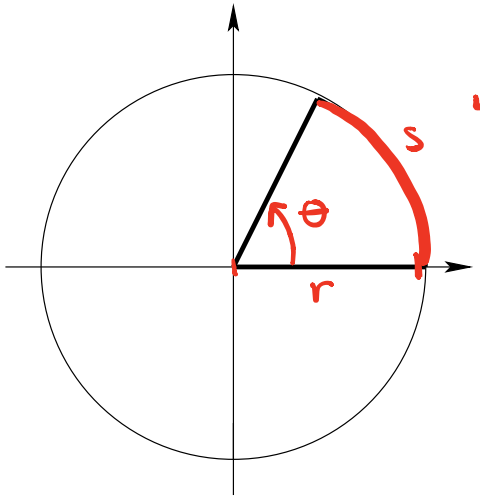
$$\boxed{\theta = \frac{s}{r}} \star$$

θ = ANGLE IN RADIAN MEASUREMENT

s = ARC LENGTH

r = RADIUS

where θ is measured in radians.



LENGTH OF ARC IS EQUAL TO THE RADIUS

$s = r$

$\theta = 1$ RADIAN

CIRCUMFERENCE $= \pi d$

$= \pi(2r)$

$= 2\pi r$

NOTE: For the angle $\theta = 1$ revolution:

The length of the arc (circumference) $s = 2\pi r$

Therefore, $\boxed{\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi.}$ $\star$

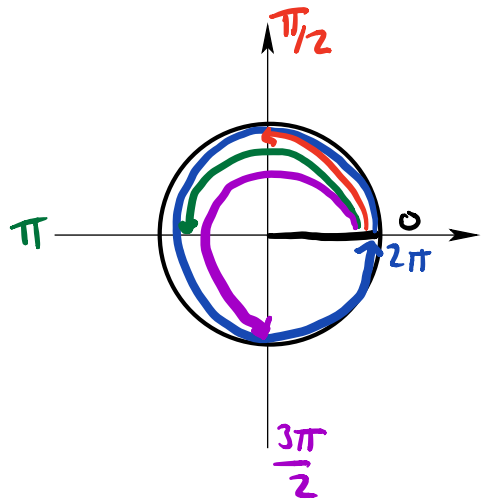
1 REVOLUTION $= 2\pi$ RADIANS

We have,

$$1 \text{ revolution} = 2\pi \text{ radians}$$

NOTE: If $0 \leq \theta \leq 2\pi$, the standard position of the angle θ in the Cartesian coordinate system is shown below:

$3\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$



NOTE: Given an angle θ, the coterminal angles to θ are

$$\theta + 2n\pi$$

where n is an integer.

$(2\pi)n = 2n\pi$

n IS AN INTEGER

ex. Find the angle with the smallest positive measure that is coterminal with $\theta = -\dfrac{21\pi}{4}$.

$n=1$: $-\frac{21\pi}{4} + 2\pi = -\frac{21\pi}{4} + \frac{8\pi}{4} = -\frac{13\pi}{4}$

$n=2$: $-\frac{21\pi}{4} + 2\pi(2) = -\frac{21\pi}{4} + \frac{16\pi}{4} = -\frac{5\pi}{4}$

$n=3$: $-\frac{21\pi}{4} + 2\pi(3) = -\frac{21\pi}{4} + \frac{24\pi}{4}$

$= \frac{3\pi}{4}$

Checkpoint: Lecture 25, problem 1

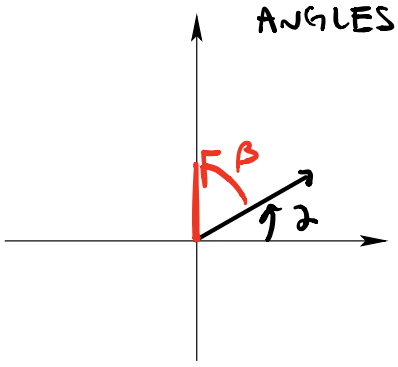
Def. Given two positive angles α and β,

1. α and β are **complementary** if $\alpha + \beta = \frac{\pi}{2}$ OR $\alpha + \beta = 90^\circ$

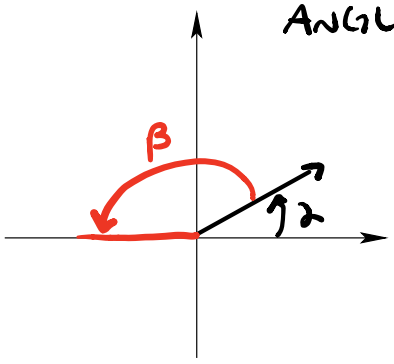
★ $\frac{\pi}{2}$ AND 90° ARE THE SAME THING!

2. α and β are **supplementary** if $\alpha + \beta = \pi$ OR $\alpha + \beta = 180^\circ$

COMPLEMENTARY ANGLES



SUPPLEMENTARY ANGLES



ex. Find the complement and supplement of the angle $\theta = \frac{\pi}{7}$. LET β = ANGLE WE ARE LOOKING FOR

COMPLEMENT:

$$\frac{\pi}{7} + \beta = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2} - \frac{\pi}{7} = \frac{7\pi}{14} - \frac{2\pi}{14} = \frac{5\pi}{14} \quad \boxed{\beta = \frac{5\pi}{14}}$$

SUPPLEMENT:

$$\frac{\pi}{7} + \beta = \pi \Rightarrow \beta = \pi - \frac{\pi}{7} = \frac{7\pi}{7} - \frac{\pi}{7} = \frac{6\pi}{7} \quad \boxed{\beta = \frac{6\pi}{7}}$$

Checkpoint: Lecture 25, problem 2

Degree Measure

Another way to measure angles is in terms of **degrees**, denoted by °.

$$1 \text{ counterclockwise revolution} = 360^\circ$$

NOTE: $\boxed{1 \text{ revolution} = 360^\circ = 2\pi \text{ rad}}$ ☆

$$\Longrightarrow \quad 180^\circ = \pi \text{ rad}$$

$[180^\circ = \pi \text{ RAD}]\frac{1}{\pi}$

$\frac{180^\circ}{\pi} = 1 \text{ RAD}$

Therefore, we have

$[180^\circ = \pi \text{ RAD}]\frac{1}{180}$

$$\boxed{1^\circ = \frac{\pi}{180} \text{ rad}} \quad \text{and} \quad 1 \text{ rad} = \frac{180^\circ}{\pi}$$

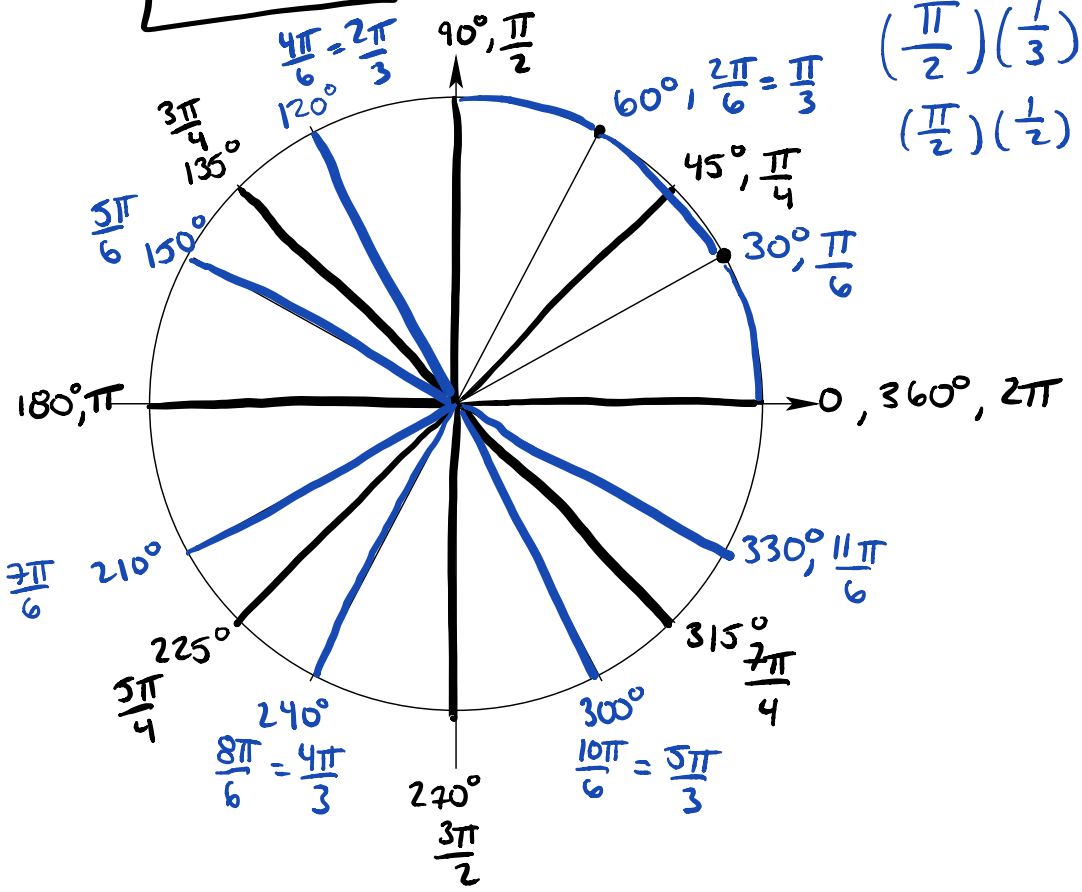
$\frac{180^\circ}{180} = \frac{\pi}{180}$

$1^\circ = \frac{\pi}{180} \text{ RAD}$

$\frac{90}{3} = 30$

$\left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right) = \frac{\pi}{6}$

$\left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right) = \frac{\pi}{4}$



Conversions between Radians and Degrees

THIS FRACTION REDUCES TO 1!

$$\text{Radian} \underset{\times \frac{\pi}{180^\circ}}{\overset{\times \frac{180^\circ}{\pi}}{\rightleftarrows}} \text{Degree}$$

ex. Convert each angle in degrees to radians.

1) $60^\circ \cdot \frac{\pi}{180^\circ} = \frac{60\pi}{180} = \frac{\pi}{3}$

2) $150^\circ \cdot \frac{\pi}{180^\circ} = \frac{150\pi}{180} = \frac{5\pi}{6}$

ex. Convert each angle in radians to degrees.

1) $\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{\cancel{\pi}180^\circ}{6\cancel{\pi}} = \frac{180^\circ}{6} = 30^\circ$

2) $-\frac{3\pi}{4} \cdot \frac{180^\circ}{\pi} = \frac{-3\cancel{\pi}(180^\circ)}{4\cancel{\pi}} = -135^\circ$

Checkpoint: Lecture 25, problem 3

"PERIMETER"

Arc Length = PARTIAL CIRCUMFERENCE AROUND CIRCLE

For a circle of radius r, a central angle θ intercepts an arc of length s given by

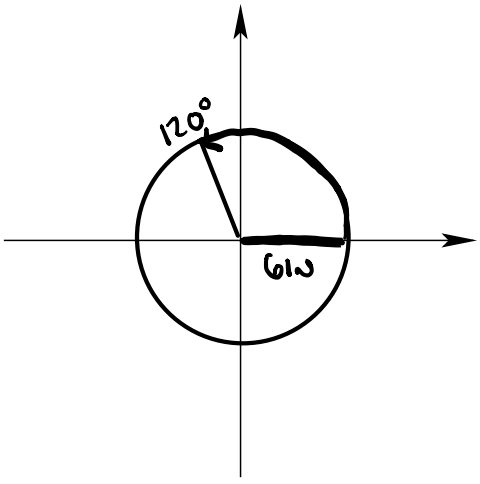
EARLIER: $\theta = \frac{s}{r}$

$$\boxed{s = r\theta}$$

s = ARC LENGTH
r = RADIUS
θ = ANGLE IN RADIANS

where θ is measured in radians.

ex. A circle has a radius of 6 inches. Find the length of the arc intercepted by a central angle of 120°.



$s = r\theta$

★WANT θ IN RADIANS!

$\theta = 120° \left(\frac{\pi}{180°}\right)$

$\theta = \frac{120\pi}{180} = \frac{2\pi}{3}$

$s = 6\left(\frac{2\pi}{3}\right) = \frac{12\pi}{3} = \boxed{4\pi \text{ INCHES}}$

Checkpoint: Lecture 25, problem 4

Area of a Sector

For a circle of radius r, the area A of a sector with central angle θ is given by

$$\boxed{A = \frac{1}{2}r^2\theta}$$

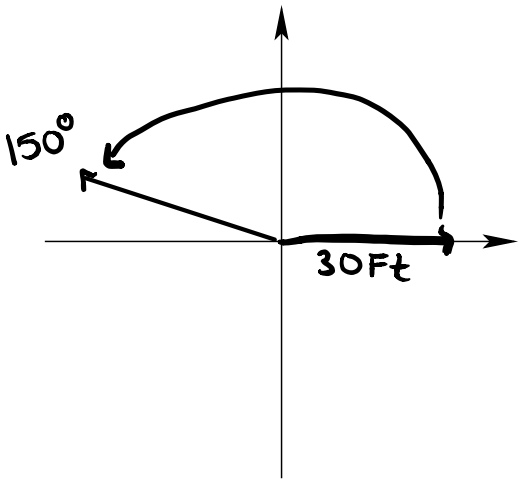
where θ is measured in radians.

$A = \pi r^2$

AREA OF A SECTOR:

$\pi r^2\left(\frac{\theta}{2\pi}\right)$

ex. A sprinkler sprays water over a distance of 30 feet while rotating through an angle of 150°. What area of lawn receives water?



WANT θ IN RADIANS!

$$\theta = 150^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{150\pi}{180} = \frac{5\pi}{6}$$

$$A = \frac{1}{2}(30^2)\left(\frac{5\pi}{6}\right)$$

$$A = \frac{30\cdot 30\cdot 5\pi}{2\cdot 6}$$

$$= \boxed{375\pi \text{ SQ.FT.}}$$

Checkpoint: Lecture 25, problem 5