

Lecture 26: Section 4.2

The Unit Circle and Trigonometric Functions

Unit circle

Derivation of the coordinates on the unit circle

Trigonometric functions

Domain, range, and periods of trigonometric functions

Even-odd properties

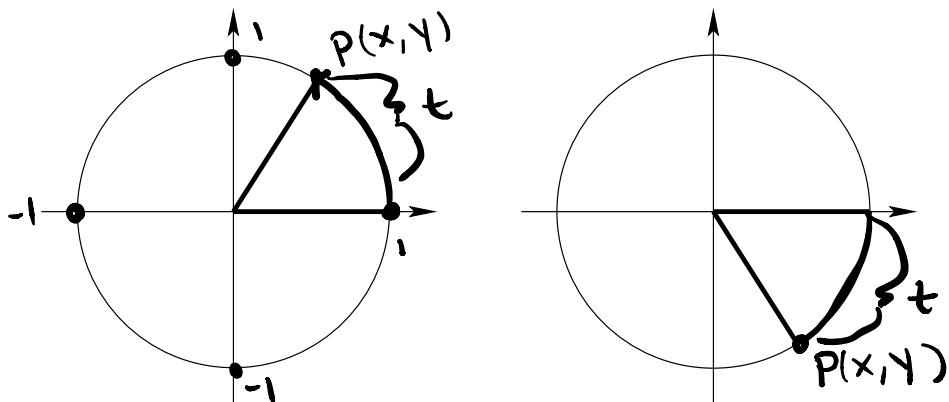
$$(h, k) \\ = (0, 0)$$

$$(x-h)^2 + (y-k)^2 = r^2 \\ x^2 + y^2 = 1$$

The **unit circle** is the circle of radius 1 centered at the origin in the xy -plane. Its equation is $x^2 + y^2 = 1$.

Let t be a real number. The **terminal point** $P(x, y)$ on the unit circle is obtained by starting at the point $(1, 0)$ and moving in a counterclockwise direction with a distance t if $t > 0$ or in a clockwise direction if $t < 0$.

t IS THE ARC LENGTH



RADIAN
MEASUREMENT

NOTE: All coterminal angles have the same terminal point.

FACT:

$$\theta = \frac{s}{r}$$

↓ ARC LENGTH
↑ RADIUS

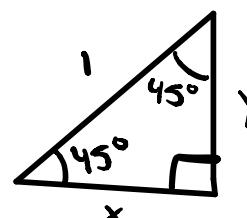
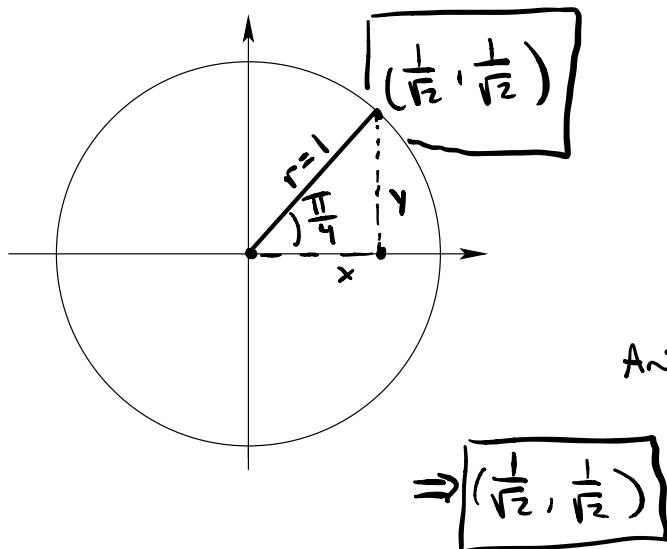
$$\text{IF } r=1, \text{ THEN } \theta = \frac{s}{1}, \text{ THUS } \theta = s$$

SO, ON A UNIT CIRCLE, THE ANGLE IN RADIAN MEASUREMENT IS EQUAL TO THE ARC LENGTH

Derivation of the Coordinates on the Unit Circle

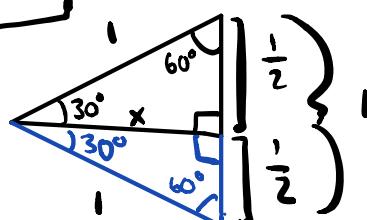
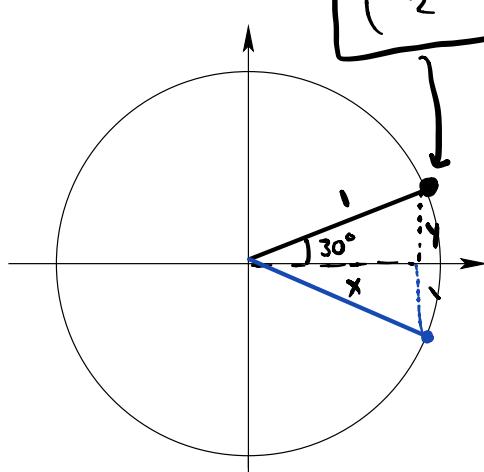
ex. Find the terminal point on the unit circle corresponding to the real number t .

$$1) t = \frac{\pi}{4} \quad \begin{matrix} \text{CUTS QUADRANT I IN HALF} \\ r = \text{RADIUS} = 1 \end{matrix}$$



$$\begin{aligned} x^2 + y^2 &= 1^2 \\ \text{NOTE: } x &= y \\ \text{B/C THIS IS} & \\ \text{AN ISOSCELES } \Delta & \\ x^2 + y^2 &= 1 \\ x^2 + x^2 &= 1 \\ 2x^2 &= 1 \\ x^2 &= \frac{1}{2} \\ x &= \pm \sqrt{\frac{1}{2}} \\ &= \pm \frac{\sqrt{1}}{\sqrt{2}} \\ &= \pm \frac{1}{\sqrt{2}} \end{aligned}$$

$$2) t = \frac{\pi}{6}$$



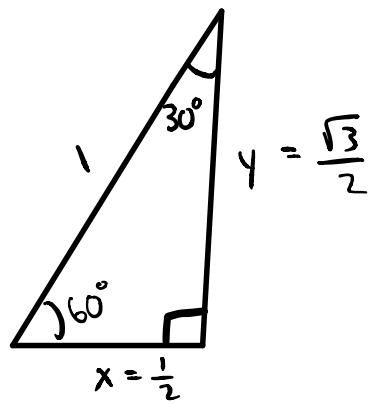
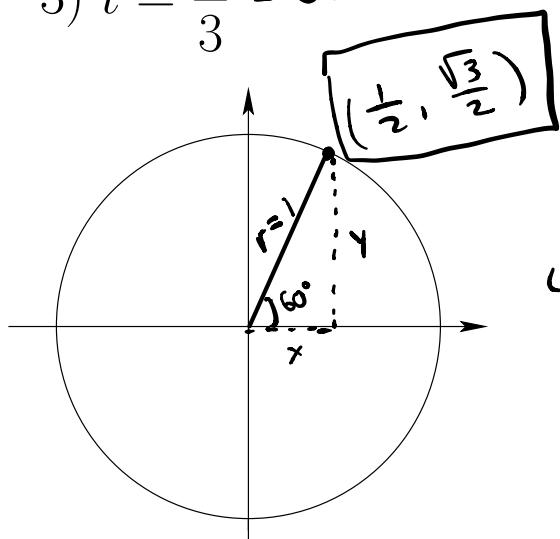
B/C EQUILATERAL $\Delta!$

L26 - 3

$$\begin{aligned} x^2 + y^2 &= 1^2 \\ x^2 + \left(\frac{1}{2}\right)^2 &= 1 \\ x^2 + \frac{1}{4} &= 1 \\ x^2 &= \frac{3}{4} \\ x &= \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2} \end{aligned}$$

$\text{QI} \Rightarrow x \text{ IS POSITIVE, SO } x = \frac{\sqrt{3}}{2}$

$$3) t = \frac{\pi}{3} = 60^\circ$$



LOOK BACK
@
PREVIOUS
EXAMPLE

Checkpoint: Lecture 26, problem 1

$$4\left(\frac{\pi}{6}\right) = \frac{4\pi}{6} = \frac{2\pi}{3}$$

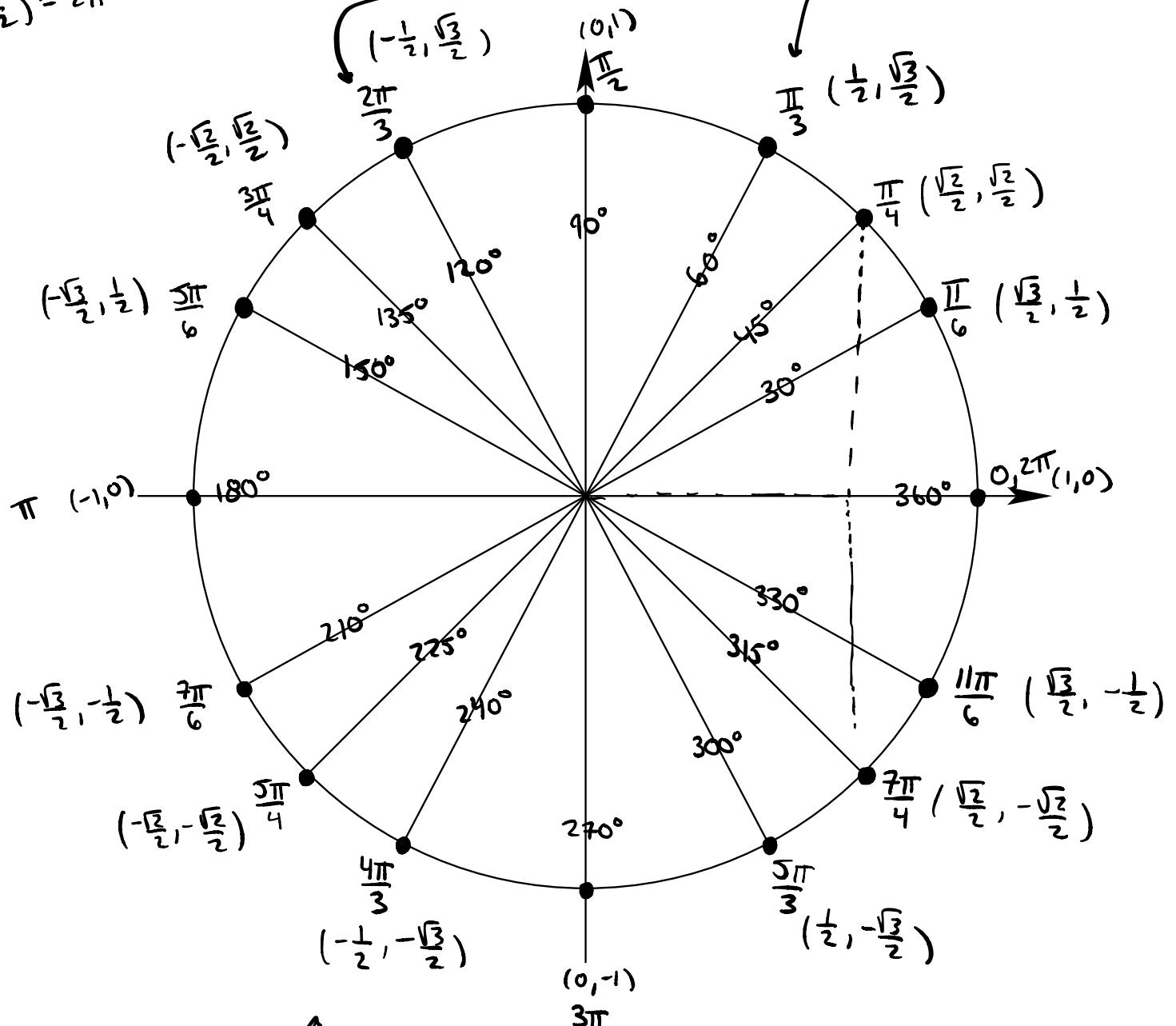
$$2\left(\frac{\pi}{6}\right) = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$2\left(\frac{\pi}{2}\right) = \pi$$

$$3\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

$$4\left(\frac{\pi}{2}\right) = 2\pi$$

The Unit Circle



$$8\left(\frac{\pi}{6}\right) = \frac{8\pi}{6} = \frac{4\pi}{3}$$

★ MEMORIZE THIS! Hint, memorize QI

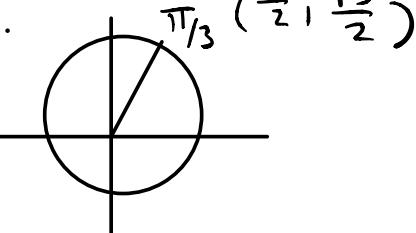
The Trigonometric Functions

Def. Let t be any real number and let (x, y) be the terminal point on the unit circle corresponding to t .

We define

SINE	$\sin t = y$	RECIPROCAL	$\csc t = \frac{1}{y}$	COSECANT
COSINE	$\cos t = x$		$\sec t = \frac{1}{x}$	SECANT
TANGENT	$\tan t = \frac{y}{x}$		$\cot t = \frac{x}{y}$	COTANGENT

ex. Find the six trigonometric functions of the given real number t .



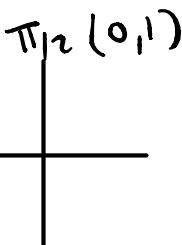
$$1) t = \frac{\pi}{3}$$

$$\sin t = \frac{\sqrt{3}}{2}$$

$$\cos t = \frac{1}{2}$$

$$\tan t = \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$$

$$2) t = \frac{\pi}{2}$$



$$\sin t = 1$$

$$\cos t = 0$$

$$\tan t = \frac{1}{0}$$

$$\csc t = \frac{1}{\frac{\sqrt{3}}{2}} = \left(\frac{2}{\sqrt{3}}\right) \frac{\sqrt{3}}{\sqrt{3}}$$

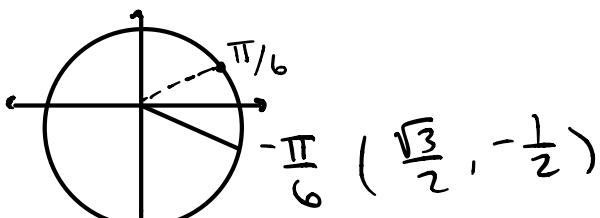
$$= \frac{2\sqrt{3}}{3}$$

$$\sec t = \frac{1}{\frac{1}{2}} = 2$$

$$\cot t = \frac{1}{\frac{\sqrt{3}}{2}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc t = \frac{1}{1} = 1$$

$$\sec t = \frac{1}{0} \text{ UNDEFINED!}$$



$$3) t = -\frac{\pi}{6}$$

$$\sin t = -\frac{1}{2}$$

$$\cos t = \frac{\sqrt{3}}{2}$$

$$\tan t = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

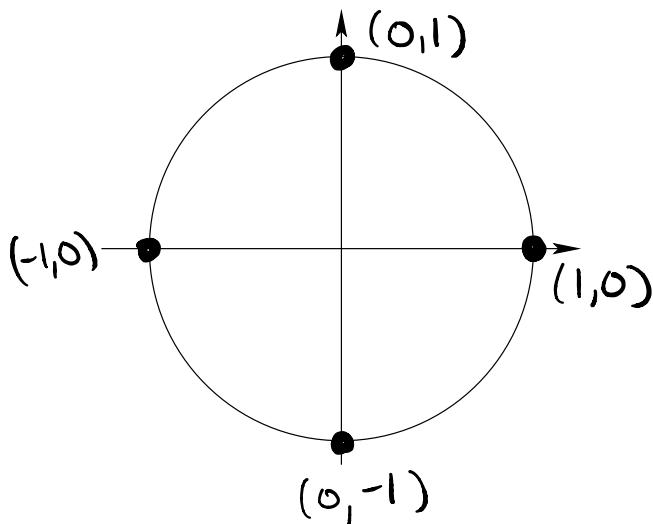
Checkpoint: Lecture 26, problem 2

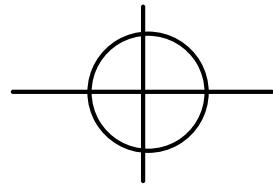
Domain and Range of Sine and Cosine

FOR ALL CIRCLES, EVEN IF
NOT A UNIT CIRCLE

Domain: all real numbers

Range: $[-1, 1]$





Periodic Functions

Def. A function f is **periodic** if there is a positive number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f . The smallest number c satisfying the above equation is called the **period** of f .

We have

$$\sin(t + 2\pi n) = \sin t \text{ and}$$

$$\cos(t + 2\pi n) = \cos t$$

for any integer n and real number t . The periods of sine and cosine functions are both 2π .

ex. Find

FIND SMALLEST POSITIVE COTERMINAL ANGLE

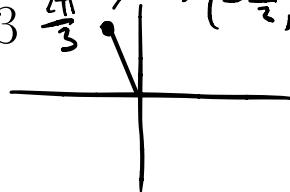
$$1) \sin\left(\frac{17\pi}{4}\right) \quad \frac{17\pi}{4} - 2\pi = \frac{17\pi}{4} - \frac{8\pi}{4} = \frac{9\pi}{4}$$

$$\frac{9\pi}{4} - 2\pi = \frac{9\pi}{4} - \frac{8\pi}{4} = \frac{\pi}{4}$$

$$2) \cos\left(-\frac{4\pi}{3}\right) \quad \Rightarrow \sin\left(\frac{17\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

Checkpoint: Lecture 26, problem 3 $\frac{4\pi}{3} \rightarrow (-\frac{1}{2}, \frac{\sqrt{3}}{2})$

$$-\frac{4\pi}{3} + 2\pi = -\frac{4\pi}{3} + \frac{6\pi}{3} = \frac{2\pi}{3}$$



$$\Rightarrow \cos\left(-\frac{4\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

EVEN: $f(-x) = f(x)$ $(x, y) \leftrightarrow (-x, y)$

ODD: $f(-x) = -f(x)$ $(x, y) \leftrightarrow (-x, -y)$

★ Even-Odd Properties

1. The cosine and secant functions are even.

$$\boxed{\cos(-t) = \cos t} \quad \boxed{\sec(-t) = \sec t}$$

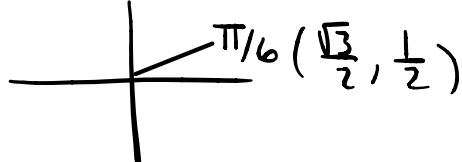
2. The sine, cosecant, tangent, and cotangent functions are odd.

$$\sin(-t) = -\sin t \quad \csc(-t) = -\csc t$$

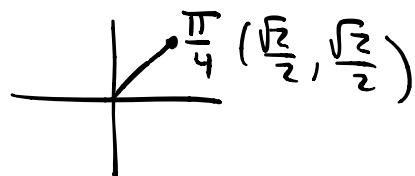
$$\tan(-t) = -\tan t \quad \cot(-t) = -\cot t$$

ex. Find

$$1) \sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$



$$2) \cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$



Checkpoint: Lecture 26, problem 4