

## **Lecture 27 Sections 4.3 and 4.4**

### **Section 4.3**

#### **Right Triangle Trigonometry**

SOH CAH TOA

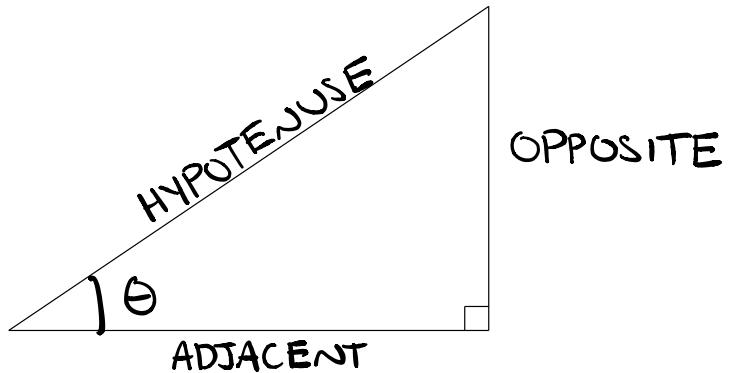
Cofunctions of complementary angles

Trigonometric identities

### **Section 4.4 Trigonometric Functions of Any Angle**

Reference angle

Angle of elevation and angle of depression



**Def.** Let  $\theta$  be an acute angle of a right triangle. The six trigonometric functions of the angle  $\theta$  are defined as follows.

RECIPROCAL

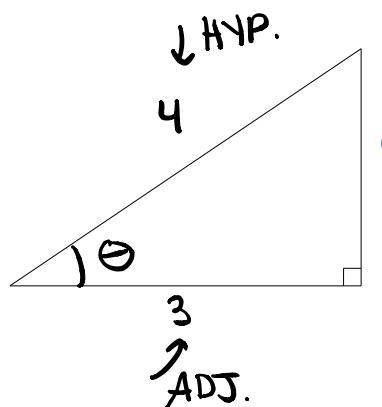
$$\sin \theta = \frac{\text{OPP.}}{\text{HYP.}} \quad \text{"SOH"} \qquad \csc \theta = \frac{\text{HYP.}}{\text{OPP.}}$$

$$\cos \theta = \frac{\text{ADJ.}}{\text{HYP.}} \quad \text{"CAH"} \qquad \sec \theta = \frac{\text{HYP.}}{\text{ADJ.}}$$

$$\tan \theta = \frac{\text{OPP.}}{\text{ADJ.}} \quad \text{"TOA"} \qquad \cot \theta = \frac{\text{ADJ.}}{\text{OPP.}}$$

S O H	C A H	T O A
I P Y	O D Y	A P D
N P P	S J P	N P J

ex. If  $\cos \theta = \frac{3}{4}$ , sketch a right triangle with acute angle  $\theta$ , and find the other values of the five trigonometric functions of  $\theta$ .



$$\cos \theta = \frac{3}{4} = \frac{\text{ADJ}}{\text{HYP}}$$

PYTHAGOREAN THEOREM

$$4^2 = 3^2 + (\text{OPP.})^2$$

$$16 = 9 + (\text{OPP.})^2$$

$$7 = (\text{OPP.})^2 \Rightarrow \text{OPP.} = \pm \sqrt{7}$$

$$\text{OPP.} = \sqrt{7}$$

$$\sin \theta = \frac{\text{OPP.}}{\text{HYP.}} = \frac{\sqrt{7}}{4}$$

$$\csc \theta = \frac{\text{HYP.}}{\text{OPP.}} = \frac{4}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$$

$$\cos \theta = \frac{3}{4}$$

$$\sec \theta = \frac{\text{HYP.}}{\text{ADJ.}} = \frac{4}{3}$$

$$\tan \theta = \frac{\text{OPP.}}{\text{ADJ.}} = \frac{\sqrt{7}}{3}$$

$$\cot \theta = \frac{\text{ADJ.}}{\text{OPP.}} = \frac{3}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$$

Checkpoint: Lecture 27, problem 1

**Theorem:** Cofunctions of complementary angles are equal. That is,

$$\sin(90^\circ - \theta) = \cos \theta$$

$$\cos(90^\circ - \theta) = \sin \theta$$

$$\tan(90^\circ - \theta) = \cot \theta$$

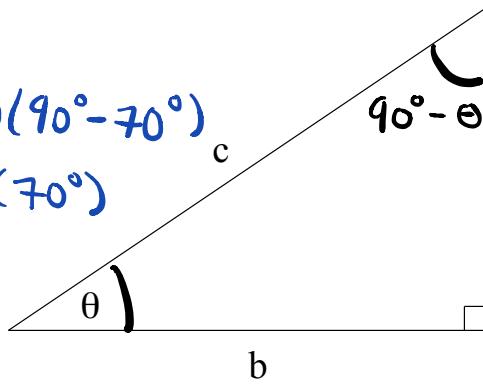
$$\cot(90^\circ - \theta) = \tan \theta$$

$$\sec(90^\circ - \theta) = \csc \theta$$

$$\csc(90^\circ - \theta) = \sec \theta$$

$$\text{Ex: } \cos(70^\circ) = \sin(20^\circ)$$

$$\begin{aligned} \text{B/C } \sin(20^\circ) &= \sin(90^\circ - 70^\circ) \\ &= \cos(70^\circ) \end{aligned}$$



$$\cos \theta = \frac{\text{ADJ.}}{\text{HYP.}} = \frac{b}{c}$$

$$^a \sin(90^\circ - \theta) = \frac{\text{OPP.}}{\text{HYP.}} = \frac{b}{c}$$

$$\Rightarrow \cos \theta = \sin(90^\circ - \theta)$$

\* ~~90-54=36~~ ex. Evaluate:

$$1) \sin 36^\circ - \cos 54^\circ$$

$$= \sin(90^\circ - 54^\circ) - \cos(54^\circ) = \cos(54^\circ) - \cos(54^\circ) = 0$$

$$2) \frac{\tan 56^\circ}{\cot 34^\circ} = \frac{\tan(90^\circ - 34^\circ)}{\cot(34^\circ)} = \frac{\cot(34^\circ)}{\cot(34^\circ)} = 1$$

Checkpoint: Lecture 27, problem 2

MUST KNOW THESE!

## Trigonometric Identities

### 1. Reciprocal Identities ~~☆☆~~

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

### 2. Quotient Identities ~~☆☆~~

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

### 3. Pythagorean Identities ~~☆☆~~

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$(\sin^2 \theta + \cos^2 \theta = 1) \frac{1}{\cos^2 \theta}$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \frac{1}{\cos^2 \theta}$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \Rightarrow 1 + \cot^2 \theta = \csc^2 \theta$$

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ex. Let  $\theta$  be an acute angle such that  $\sin \theta = \frac{1}{3}$ .  
Use trigonometric identities to find

1)  $\cos \theta$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\left(\frac{1}{3}\right)^2 + \cos^2 \theta = 1$$

$$\frac{1}{9} + \cos^2 \theta = 1$$

$$\cos^2 \theta = 1 - \frac{1}{9}$$

$$\cos^2 \theta = \frac{8}{9}$$

$$\cos \theta = \sqrt{\frac{8}{9}}$$

$$= \frac{\sqrt{8}}{\sqrt{9}} = \frac{\sqrt{4 \cdot 2}}{3}$$

ACUTE  
ANGLE  
⇒ ONLY  
POSITIVE  
VALUES!

2)  $\tan \theta$  COULD USE  $\tan^2 + 1 = \sec^2 \theta$   $= \frac{2\sqrt{2}}{3}$

OR, QUICKER:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{1}{3}}{\frac{2\sqrt{2}}{3}} = \frac{1}{3} \cdot \frac{3}{2\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

Checkpoint: Lecture 27, problem 3

## Lecture 27, Part II: Section 4.4

### Trigonometric Functions of Any Angle

NOT NECESSARILY A UNIT CIRCLE WHERE  
**Def.** Let  $\theta$  be an angle in standard position and let  $r = 1$   
 $(x, y)$  be a point on the terminal side. If  
 $r = \sqrt{x^2 + y^2}$  is the distance from the origin to the  
point  $(x, y)$ , then



$$\sin \theta = \frac{y}{r}$$

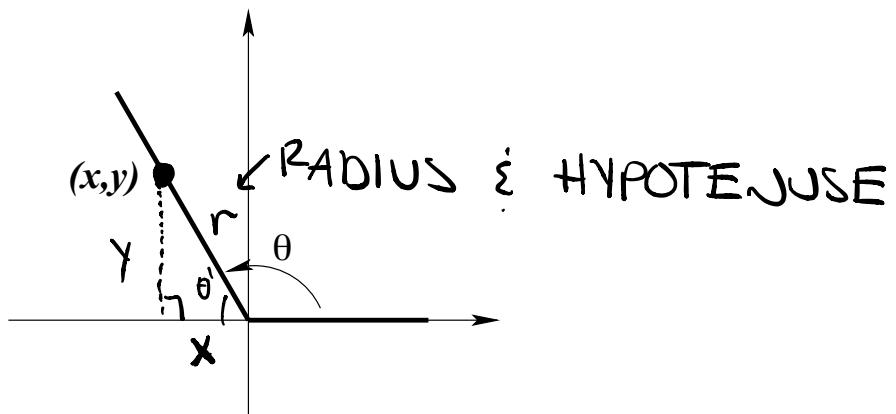
$$\csc \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r}$$

$$\sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x}$$

$$\cot \theta = \frac{x}{y}$$



**NOTE:** If  $x = 0$ ,  $\tan \theta$  and  $\sec \theta$  are undefined. If  $y = 0$ ,  $\cot \theta$  and  $\csc \theta$  are undefined.

ex. Given point  $(-5, 12)$  on the terminal side of the angle  $\theta$ , find the six trigonometric function values of the angle  $\theta$ .  $r = \sqrt{(-5-0)^2 + (12-0)^2}$

$$\begin{aligned} &= \sqrt{25+144} \\ &= \sqrt{169} = 13 \end{aligned}$$

$$\sin \theta = \frac{y}{r} = \frac{12}{13}$$

$$\csc \theta = \frac{r}{y} = \frac{13}{12}$$

$$\cos \theta = \frac{x}{r} = \frac{-5}{13}$$

$$\sec \theta = \frac{r}{x} = -\frac{13}{5}$$

$$\tan \theta = \frac{y}{x} = \frac{12}{-5}$$

$$\cot \theta = \frac{x}{y} = \frac{-5}{12}$$

ex. If  $\cos \theta = \frac{3}{5}$  and  $\theta$  lies in Quadrant IV, find the values of all the trigonometric functions.

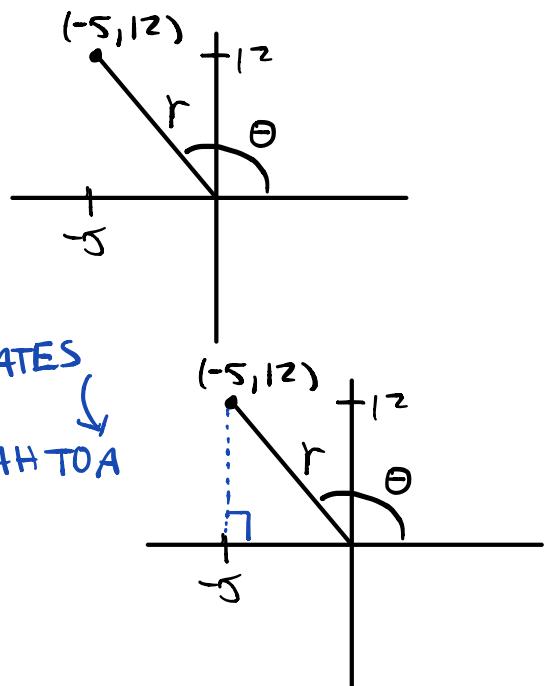
$$\cos \theta = \frac{x}{r} = \frac{3}{5} \quad \sec \theta = \frac{r}{x} = \frac{5}{3}$$

$$\sin \theta = \frac{y}{r} = -\frac{4}{5}$$

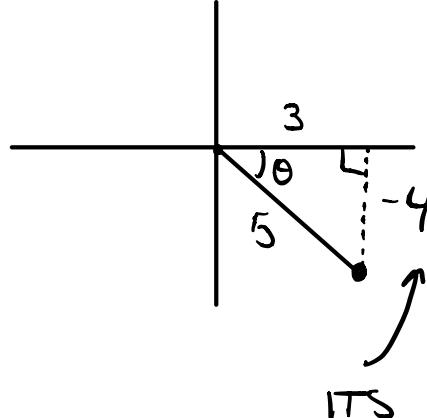
$$\csc \theta = \frac{r}{y} = -\frac{5}{4}$$

$$\tan \theta = \frac{y}{x} = -\frac{4}{3}$$

$$\cot \theta = \frac{x}{y} = -\frac{3}{4}$$



Checkpoint: Lecture 27, problem 4



$$3^2 + y^2 = 5^2$$

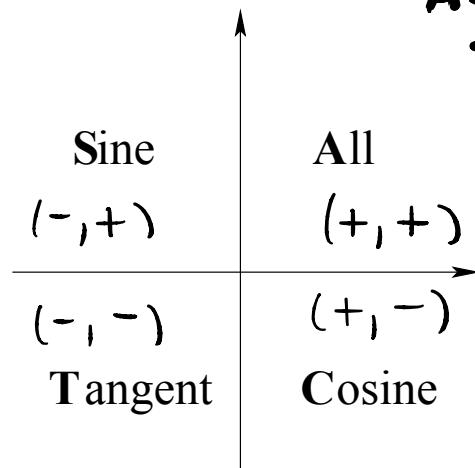
$$y^2 = 25 - 9$$

$$y^2 = 16$$

$$y = -4$$

## Signs of the Trigonometric Functions

**"ALL STUDENTS  
TAKE CALCULUS"**

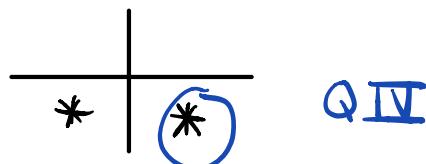


ex. Find the quadrant in which  $\theta$  lies if  $\tan \theta < 0$   
and  $\sin \theta < 0$

$$\tan \theta = \frac{y}{x} = \frac{-}{+}$$

$y$  IS NEGATIVE

$x$  IS POSITIVE

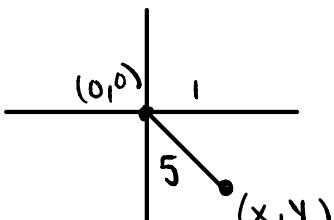


ex. Given  $\sec \theta = 5$  and  $\sin \theta < 0$ , find  $\csc \theta$ .

$$1. \sec \theta = 5 = \frac{5}{1} = \frac{r}{x} \Rightarrow x=1, r=5$$

ALSO,  $y$  IS NEGATIVE (SINCE  $\sin \theta < 0$ )

$\Rightarrow$  WE ARE IN QIV



$$d=r=5=\sqrt{(1-0)^2+(y-0)^2}$$

$$5=\sqrt{1+y^2}$$

$$25=1+y^2$$

$$24=y^2$$

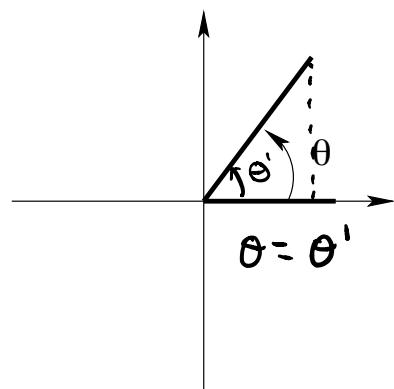
$$\pm\sqrt{24}=y \Rightarrow y=-\sqrt{24}$$

$$\csc \theta = \frac{r}{y} = \frac{5}{-2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{5\sqrt{6}}{-2\cdot 6} = \frac{5\sqrt{6}}{-12}$$

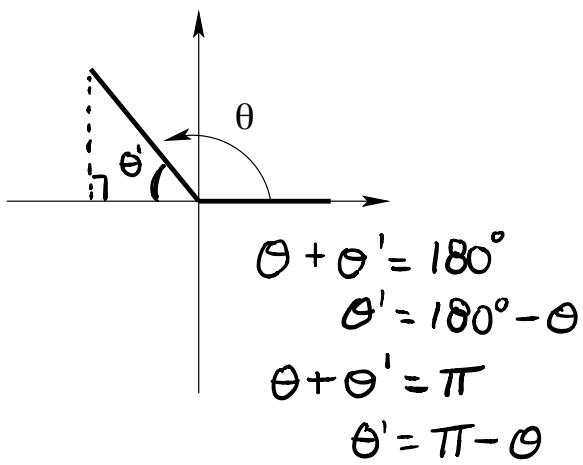
$$y=-2\sqrt{6}$$

## Reference Angles

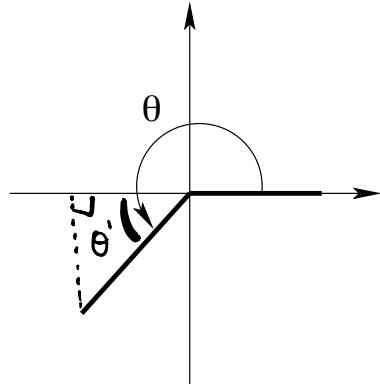
**Def.** Let  $\theta$  be an angle in standard position. The **reference angle** is the acute angle  $\theta'$  formed by the terminal side of  $\theta$  and the  $x$ -axis.



$$\theta = \theta'$$



$$\begin{aligned}\theta + \theta' &= 180^\circ \\ \theta' &= 180^\circ - \theta \\ \theta + \theta' &= \pi \\ \theta' &= \pi - \theta\end{aligned}$$



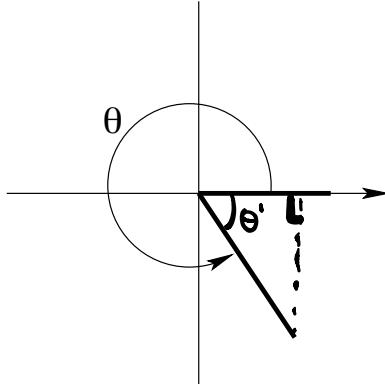
$$\theta = \pi + \theta'$$

$$\theta - \pi = \theta'$$

OR

$$\theta = 180^\circ + \theta'$$

$$\theta - 180^\circ = \theta'$$



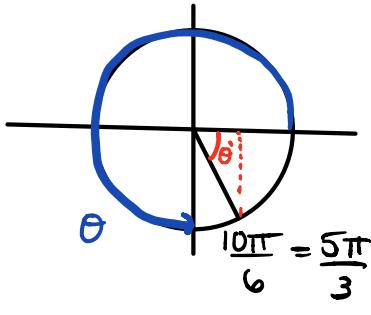
$$\begin{aligned}\theta + \theta' &= 2\pi \\ \theta' &= 2\pi - \theta\end{aligned}$$

$$\theta + \theta' = 360^\circ$$

$$\theta' = 360^\circ - \theta$$

**ex.** Find the reference angle for

$$1) \theta = \frac{5\pi}{3}$$



$$\theta + \theta' = 2\pi$$

$$\theta' = 2\pi - \theta$$

$$\theta' = 2\pi - \frac{5\pi}{3}$$

$$\theta' = \frac{6\pi}{3} - \frac{5\pi}{3}$$

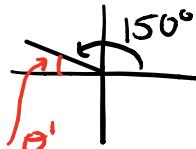
$$\theta' = \frac{\pi}{3}$$

$$2) \theta = 870^\circ$$

FIND SMALLEST POSITIVE COTERMINAL ANGLE THAT IS LESS THAN  $360^\circ$

$$870^\circ - 360^\circ = 510^\circ$$

$$510^\circ - 360^\circ = 150^\circ$$



$$180^\circ = \theta + \theta'$$

$$\Rightarrow \theta' = 180^\circ - \theta$$

$$\theta' = 180^\circ - 150^\circ$$

$$\theta' = 30^\circ$$

Checkpoint: Lecture 27, problem 5

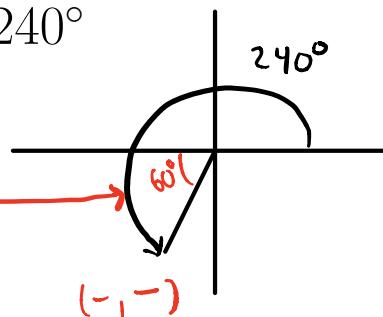
## Evaluating Trigonometric Functions for Any Angle

- Find the reference angle  $\theta'$  associated with the angle  $\theta$ .
- Determine the sign of the trigonometric function of  $\theta$  by noting the quadrant in which  $\theta$  lies.
- The value of the trigonometric function of  $\theta$  is the same, except possibly for sign, as the value of the trigonometric function of  $\theta'$ .

ex. Use the reference angle to find

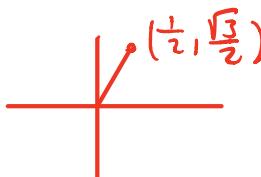
1)  $\sin 240^\circ$

$$\begin{aligned}\theta &= 180^\circ + \theta' \\ 240^\circ &= 180^\circ + \theta' \\ \Rightarrow \theta' &= 60^\circ\end{aligned}$$

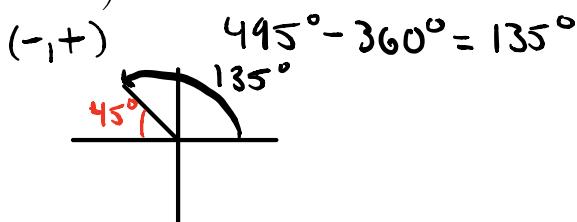


$$\sin 240^\circ = -\sin 60^\circ$$

$$= -\frac{\sqrt{3}}{2}$$



2)  $\cot 495^\circ$



$$\cot 495^\circ = \cot 135^\circ$$

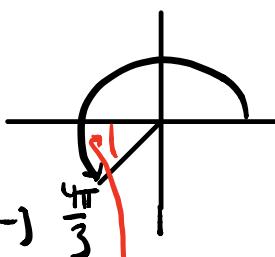
$$= -\cot 45^\circ$$

$$\begin{aligned}\theta + \theta' &= 180^\circ \\ \theta' &= 180^\circ - \theta = 180^\circ - 135^\circ = 45^\circ\end{aligned}$$

3)  $\sin\left(\frac{16\pi}{3}\right)$

$$\frac{16\pi}{3} - 2\pi = \frac{16\pi}{3} - \frac{6\pi}{3} = \frac{10\pi}{3}$$

$$\frac{10\pi}{3} - 2\pi = \frac{10\pi}{3} - \frac{6\pi}{3} = \frac{4\pi}{3}$$

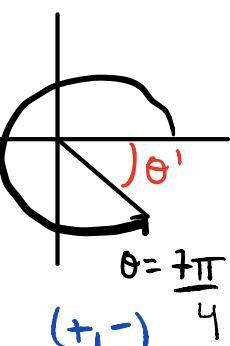


$$\sin\left(\frac{16\pi}{3}\right) = \sin\left(\frac{4\pi}{3}\right)$$

$$= -\sin\left(\frac{\pi}{3}\right) \\ = -\frac{\sqrt{3}}{2}$$

4)  $\sec\left(-\frac{\pi}{4}\right)$

$$-\frac{\pi}{4} + 2\pi = -\frac{\pi}{4} + \frac{8\pi}{4} = \frac{7\pi}{4}$$



$$\theta' + \theta = 2\pi$$

$$\theta' + \frac{7\pi}{4} = 2\pi \Rightarrow \theta' = \frac{8\pi}{4} - \frac{7\pi}{4} = \frac{\pi}{4}$$

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$$= \sec\left(\frac{\pi}{4}\right) = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$



$$= \frac{2\sqrt{2}}{2} \\ = \sqrt{2}$$

ex. If  $\tan \theta = \frac{2}{3}$  and  $\theta$  is in Quadrant III, find  $\cos \theta$ .

**Method 1:** Use trigonometric identities.

$$\tan^2 \theta + 1 = \sec^2 \theta$$

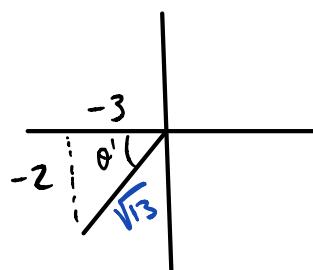
$$\left(\frac{2}{3}\right)^2 + 1 = \sec^2 \theta$$

$$\frac{13}{9} = \sec^2 \theta$$

$$\sec(\theta) = \pm \sqrt{\frac{13}{9}} = -\frac{\sqrt{13}}{3}$$

$$\Rightarrow \cos \theta = \frac{1}{\sec \theta} = -\frac{1}{\frac{\sqrt{13}}{3}} = -\frac{3}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$$

**Method 2:** Use the reference angle.



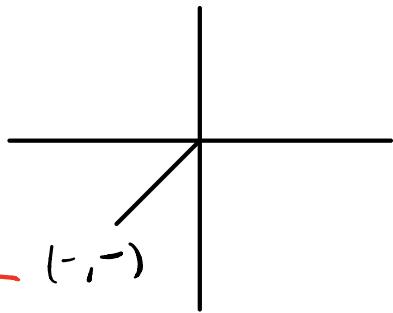
$$\tan \theta = \frac{2}{3} = \frac{\text{OPP}}{\text{ADJ}}$$

$$\cos \theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{-3}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$$

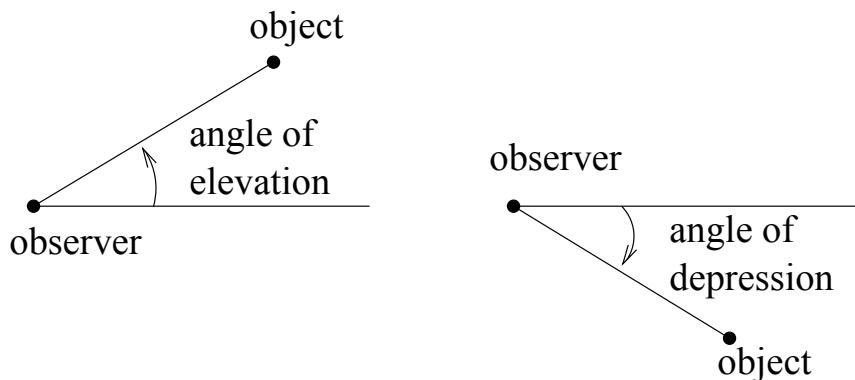
$$(-2)^2 + (-3)^2 = h^2$$

$$13 = h^2 \Rightarrow h = \sqrt{13}$$

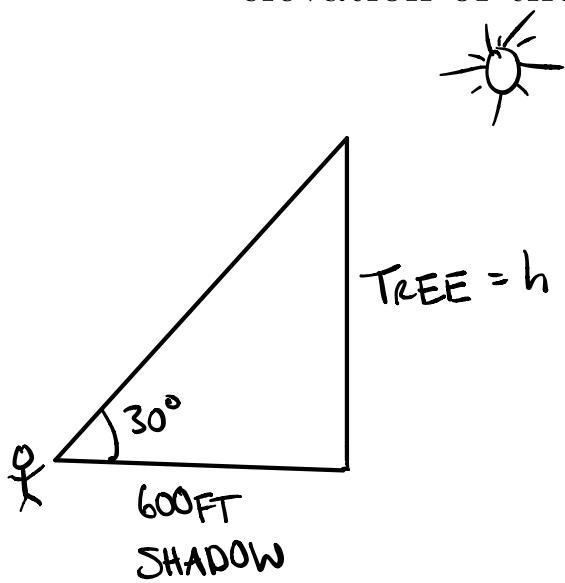
Checkpoint: Lecture 27, problem 6



# Applications



ex. A giant redwood tree casts a shadow 600 feet long. Find the height of the tree if the angle of elevation of the sun is  $30^\circ$ .

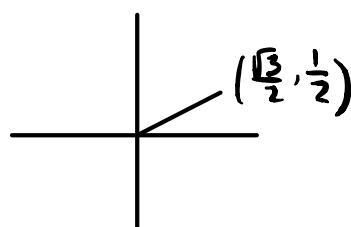


$$\tan 30^\circ = \frac{\text{OPP}}{\text{ADJ}} = \frac{h}{600}$$

$$h = 600 \tan 30^\circ$$

$$h = 600 \left( \frac{\sqrt{3}}{3} \right)$$

$$h = 200\sqrt{3} \text{ FT}$$



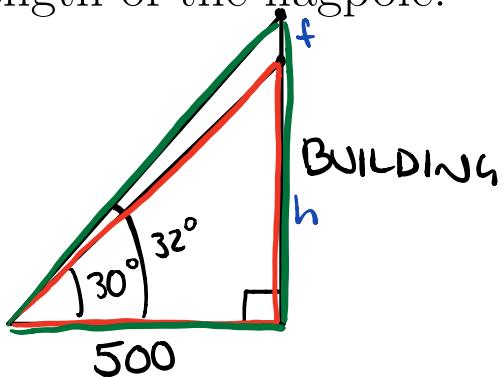
$$\tan 30^\circ = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2} \cdot \frac{2}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}$$

**ex.** From a point on the ground 500 feet from the base of a building, an observer finds that the angle of elevation to the top of the building is  $30^\circ$  and that the angle of elevation to the top of a flagpole atop the building is  $32^\circ$ . Find the height of the building and the length of the flagpole.



START WITH SMALLER  $\Delta$

$$\frac{h}{500} = \frac{\text{OPP}}{\text{ADJ}} = \tan 30^\circ$$

$$\frac{h}{500} = \tan 30^\circ$$

$$h = 500 \tan 30^\circ$$

$$h = 500 \left( \frac{\sqrt{3}}{3} \right)$$

$$h = \frac{500\sqrt{3}}{3} \text{ ft.}$$

$$f = 500 \tan(32^\circ) - h$$

→ SUBSTITUTE  $h$

$$f = 500 \tan(32^\circ) - \frac{500\sqrt{3}}{3} \text{ ft.}$$