Lecture 28: Section 4.5
Graphs of Sine and Cosine Functions

Period of sine and cosine

Amplitude of sine and cosine

Horizontal translation - phase shift

Vertical translations

THINK ABOUT THE UNIT CIRCLE

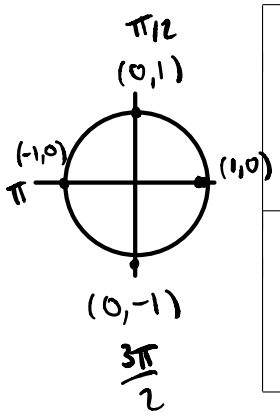
The Graph of $y = \sin x$

Domain: $(-\infty, \infty)$ ← ANY X-VALUE YOU WANT

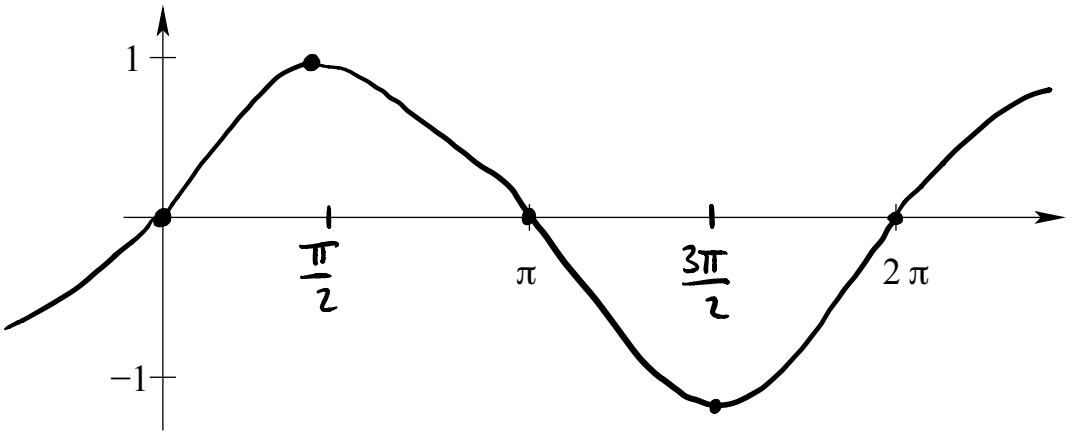
Range: $[-1, 1]$

Period: 2π (STARTS REPEATING THE CYCLE AFTER 2π)

The key points in one period:



x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin x$	0	1	0	-1	0



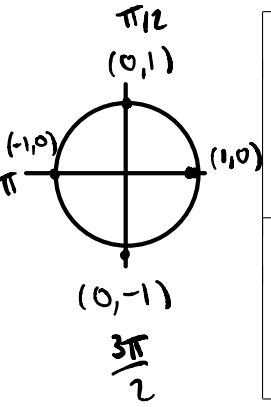
The Graph of $y = \cos x$

Domain: $(-\infty, \infty)$ ANY X-VALUE YOU WANT!

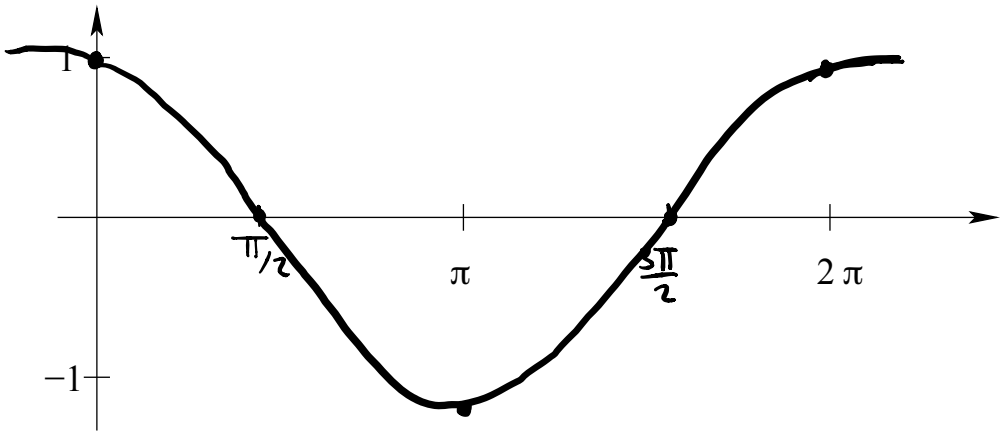
Range: $[-1, 1]$

Period: 2π

The key points in one period:



x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \cos x$	1	0	−1	0	1



Checkpoint: Lecture 28, problem 1

Amplitude for Functions $y = a\sin x$ and $y = a\cos x$

Def. The **amplitude** of $y = a\sin x$ and $y = a\cos x$ represents half the distance between the maximum and minimum values of the function and is given by

$$\text{Amplitude} = |a| = \frac{\text{maximum value - minimum value}}{2}$$

NOTE:

1. If $|a| > 1$, the curve is stretched vertically, and if $|a| < 1$, the curve is shrunk vertically.

2. The range is $[-a, a]$.

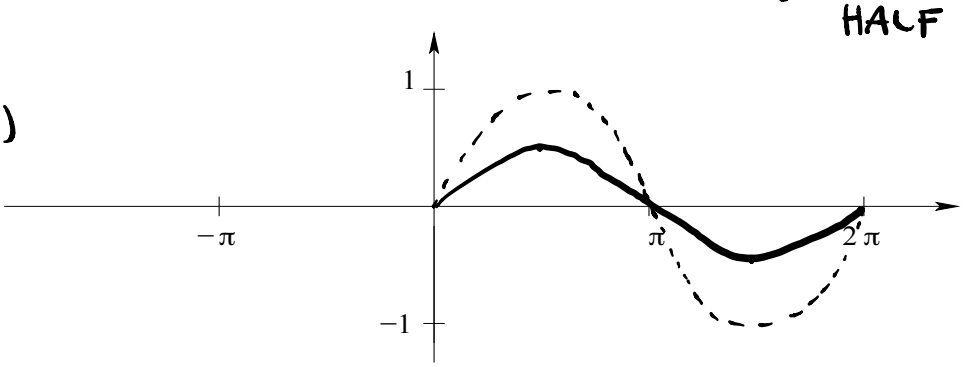
1. $y = \sin(x)$
2. $y = \frac{1}{2}\sin(x)$
↗ CUT EACH y-VALUE IN HALF

ex. Graph $y = \frac{1}{2}\sin x$

AMP. $= \left|\frac{1}{2}\right| = \frac{1}{2}$

DOMAIN: $(-\infty, \infty)$

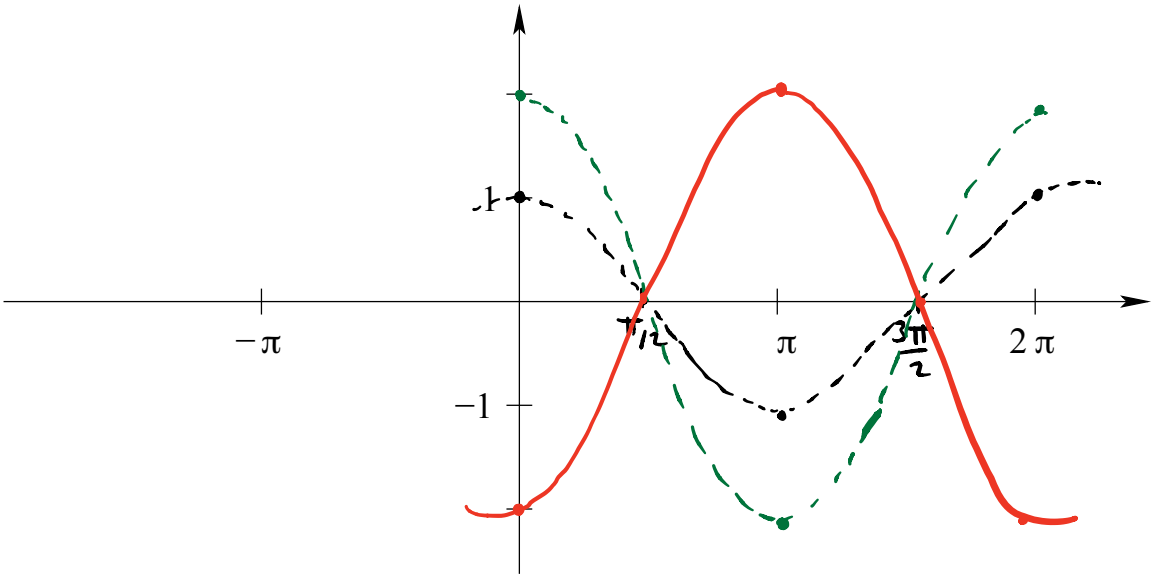
RANGE: $\left[-\frac{1}{2}, \frac{1}{2}\right]$



MULTIPLY EACH Y-VALUE BY 2

1. $y = \cos(x)$
2. $y = 2\cos(x)$
3. $y = -2\cos(x)$

ex. Graph $y = -2\cos x$



AMP. $= |-2| = 2$

OR: AMP. $= \frac{\text{MAX} - \text{MIN}}{2} = \frac{2-(-2)}{2} = \frac{4}{2} = 2$

DOMAIN: $(-\infty, \infty)$

RANGE: $[-2, 2]$

Checkpoint: Lecture 28, problem 2

Period for Functions $y = a\sin(bx)$ and $y = a\cos(bx)$

The **period** of $y = a\sin(bx)$ and $y = a\cos(bx)$ is given by

$y = \sin x$ HAS A NORMAL PERIOD AS x GOES FROM 0 TO 2π.

$$\text{Period} = \frac{2\pi}{b}$$

★ IMPORTANT!

↙ NEW PERIOD

$bx = 0$ ↓ NORMAL START

$x = 0$

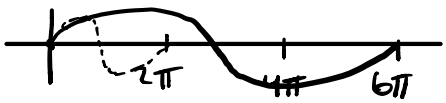
where b is a positive real number.

$bx = 2\pi$ ↙ NORMAL END

$x = \frac{2\pi}{b}$

PER. $= \frac{2\pi}{1/3} = 6\pi$

$y = \sin(\frac{1}{3}x)$



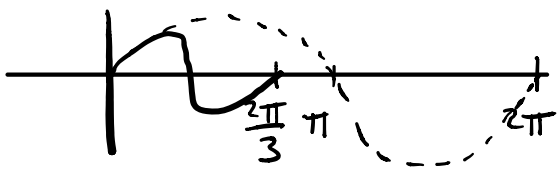
NOTE:

1. If $0 < b < 1$, the period of $y = a\sin(bx)$ is greater than 2π and represents a horizontal stretching of the graph $y = a\sin x$.

2. If $b > 1$, the period of $y = a\sin(bx)$ is less than 2π and represents a horizontal shrinking of the graph $y = a\sin x$.

$y = \sin(3x)$

PER $= \frac{2\pi}{3}$

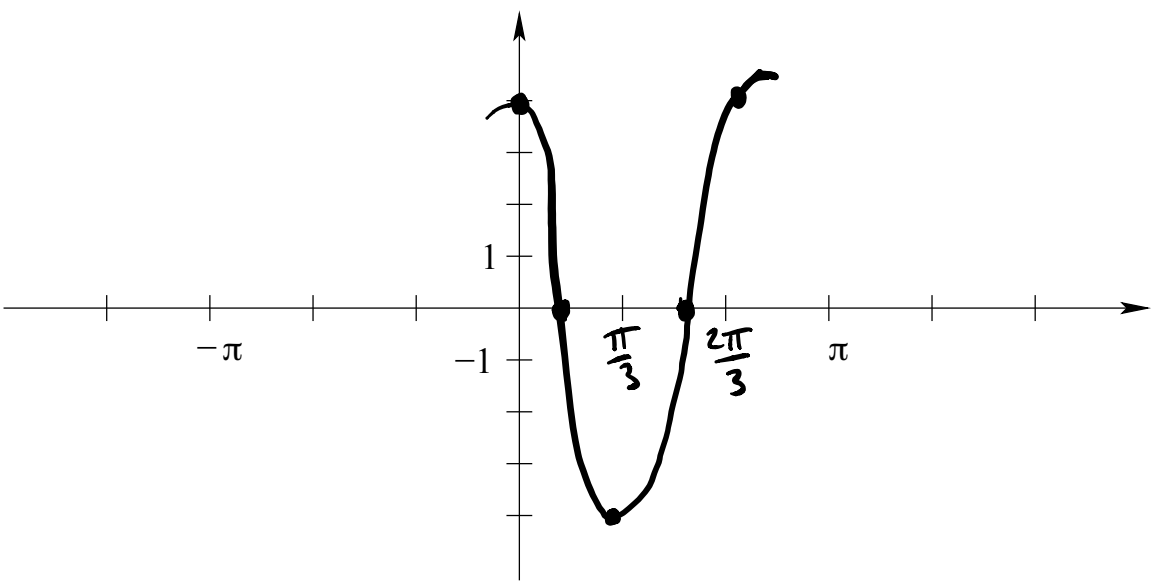


ex. Find the amplitude and period of each function, and sketch its graph.

AMP $= |4| = 4$

PER $= \frac{2\pi}{b} = \frac{2\pi}{3}$

1) $y = 4\cos(3x)$



$y = 4\cos(3x)$

$3x = 0$

$x = 0$ ↙ NEW START

$3x = 2\pi$

$x = \frac{2\pi}{3}$ ↙ NEW END

$\frac{0+\frac{\pi}{3}}{2} = \frac{\pi}{3}\cdot\frac{1}{2} = \pi/6$

$\frac{0+\frac{2\pi}{3}}{2} = \frac{2\pi}{3}\cdot\frac{1}{2} = \frac{2\pi}{6} = \frac{\pi}{3}$

X	0	π/6	π/3	π/2	2π/3
Y	4	0	−4	0	4

NORMAL Y-VALUES → 1, 0, −1, 0, 1

$4\cdot 1 = 4$, $4\cdot 0 = 0$, $4\cdot -1 = -4$, $4\cdot 0 = 0$, $4\cdot 1 = 4$

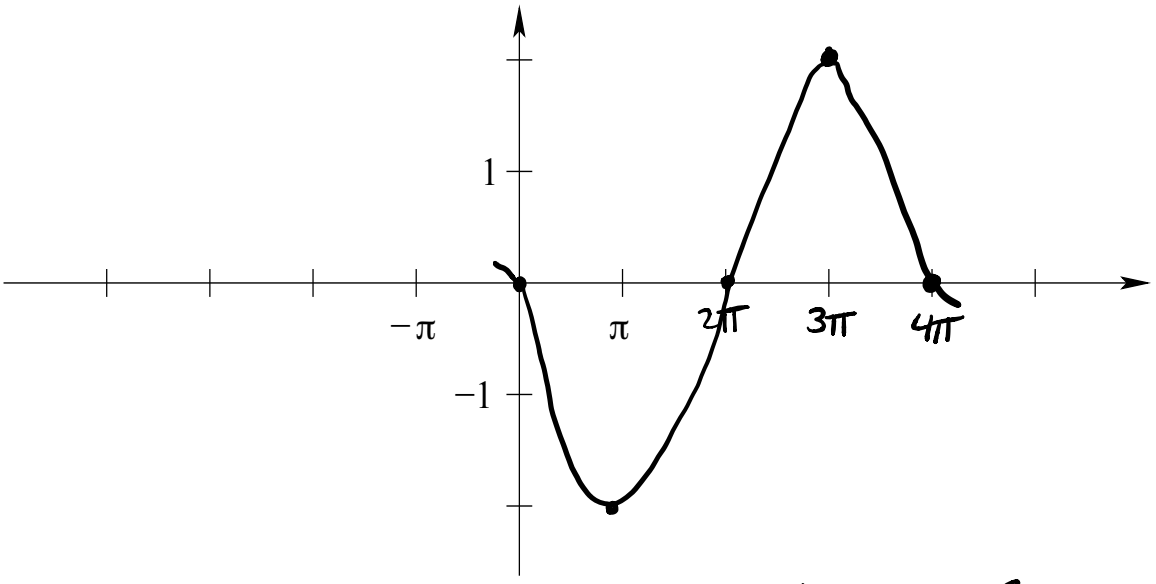
$\frac{\frac{\pi}{3}+\frac{2\pi}{3}}{2} = \frac{3\pi}{3}\cdot\frac{1}{2} = \frac{3\pi}{6} = \frac{\pi}{2}$

X	0	π/2	π	3π/2	2π
Y	1	0	−1	0	1

AMP = $|-2| = 2$

PERIOD = $\frac{2\pi}{1/2} = \frac{2\pi}{1} \cdot \frac{2}{1} = 4\pi$

2) $y = -2\sin\left(\frac{1}{2}x\right)$



$\frac{1}{2}(0+4\pi) = 2\pi$

NEW START:

$\frac{1}{2}x = 0$

$x = 0$

NEW END:

$\frac{1}{2}x = 2\pi$

$x = 4\pi$

x	0	π	2π	3π	4π
y	0	-2	0	2	0
	0	1	0	-1	0

y-VALUES FOR $y = \sin(x)$

Checkpoint: Lecture 28, problem 3

Horizontal Translations ≈ PHASE SHIFT

	$y = a\sin(bx)$	$y = a\sin(bx - c)$
Amplitude	$A = \lvert a \rvert$	$A = \lvert a \rvert$
Period	$P = \frac{2\pi}{b}$	$P = \frac{2\pi}{b}$
One cycle	$0 \le bx \le 2\pi$ $0 \le x \le \frac{2\pi}{b}$	$0 \le bx - c \le 2\pi$ $\frac{c}{b} \le x \le \frac{2\pi}{b} + \frac{c}{b}$

↑ NEW START ↑ NEW END

NOTE: The graph of $y = a\sin(bx)$ is shifted by $\frac{c}{b}$ units to get the graph of $y = a\sin(bx - c)$. The number $\frac{c}{b}$ is called the **phase shift**.

MINUS SIGN

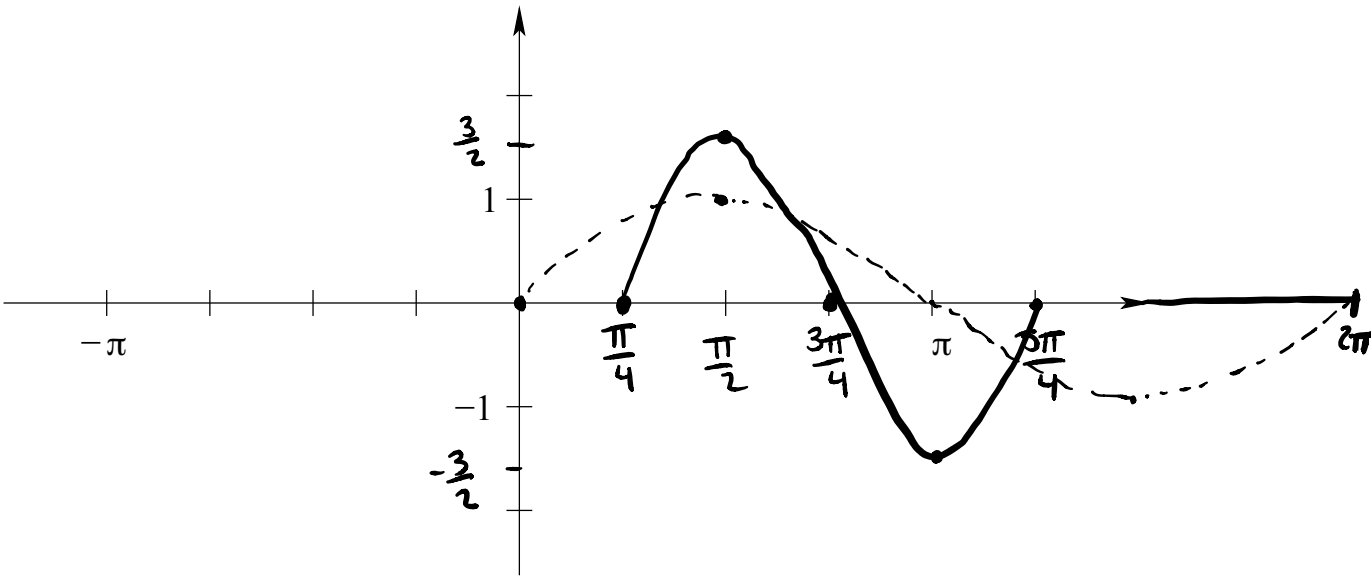
ex. Find the amplitude, period and phase shift of each function, and sketch its graph.

AMP $= \left|\frac{3}{2}\right| = \frac{3}{2}$ PERIOD $= \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$ P.S. $= \frac{c}{b}$

$= \frac{\frac{\pi}{2}}{2}$

$= \frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4}$

1) $y = \frac{3}{2}\sin\left(2x - \frac{\pi}{2}\right)$



NEW START:

$2x - \frac{\pi}{2} = 0$

$x = \frac{\pi}{4}$

$\frac{1}{2}\left(\frac{\pi}{4} + \frac{3\pi}{4}\right) = \frac{1}{2}\left(\frac{4\pi}{4}\right) = \frac{\pi}{2}$ $\quad \frac{1}{2}\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{6\pi}{4}\right) = \frac{3\pi}{4}$

x	$\pi/4$	$\pi/2$	$\frac{3\pi}{4}$	π	$5\pi/4$
y	0	$\frac{3}{2}$	0	$-\frac{3}{2}$	0
	0	1	0	-1	0

NEW END:

$2x - \frac{\pi}{2} = 2\pi$

$2x = \frac{4\pi}{2} + \frac{\pi}{2}$

$2x = \frac{5\pi}{2} \Rightarrow x = \frac{5\pi}{4}$

$\frac{1}{2}\left(\frac{5\pi}{4} + \frac{3\pi}{4}\right)$

$= \frac{1}{2}\left(\frac{8\pi}{4}\right)$

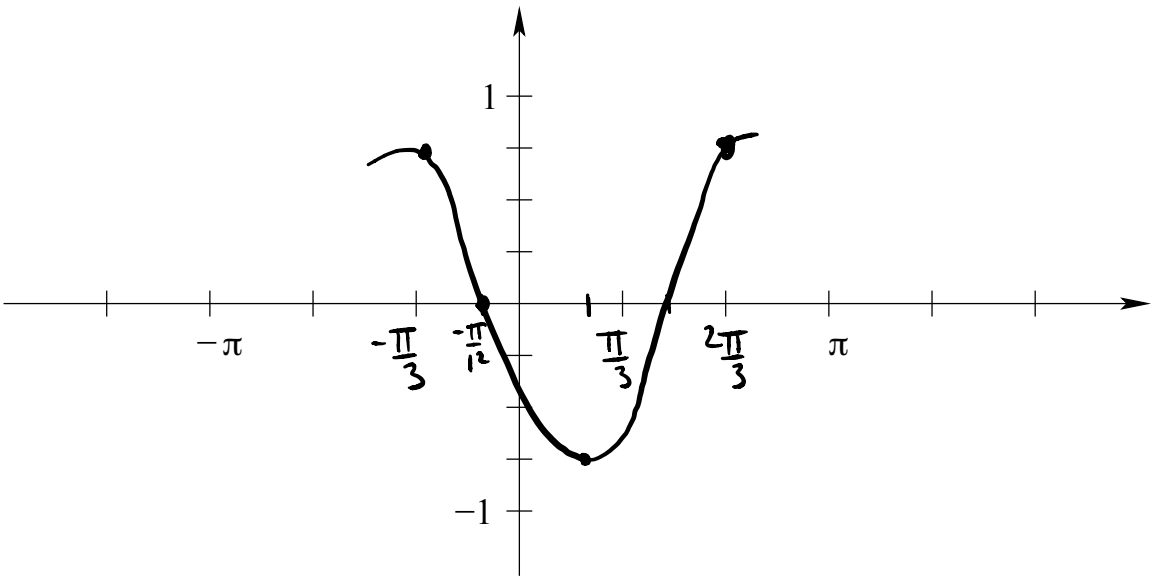
$= \pi$

AMP $= \left|\frac{3}{4}\right| = \frac{3}{4}$ PERIOD $= \frac{2\pi}{2} = \pi$

PHASE SHIFT $= \frac{c}{b} = \frac{-\frac{2\pi}{3}}{2}$

$= -\frac{2\pi}{3} \cdot \frac{1}{2} = -\frac{\pi}{3}$

2) $y = \frac{3}{4}\cos\left(2x + \frac{2\pi}{3}\right)$



NEW START:

$2x + \frac{2\pi}{3} = 0$

$2x = -\frac{2\pi}{3}$

$x = -\frac{\pi}{3}$

NEW END:

$2x + \frac{2\pi}{3} = 2\pi$

$2x = \frac{6\pi}{3} - \frac{2\pi}{3}$

$2x = \frac{4\pi}{3}$

$x = \frac{4\pi}{3} \cdot \frac{1}{2} = \frac{2\pi}{3}$

X	$-\frac{\pi}{3}$	$-\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{5\pi}{12}$	$\frac{2\pi}{3}$
Y	$\frac{3}{4}$	0	$-\frac{3}{4}$	0	$\frac{3}{4}$
	1	0	−1	0	1

$\frac{1}{2}\left(-\frac{\pi}{3} + \frac{2\pi}{3}\right) = \frac{1}{2}\left(\frac{\pi}{3}\right) = \frac{\pi}{6}$

$\frac{1}{2}\left(-\frac{\pi}{3} + \frac{\pi}{6}\right) = \frac{1}{2}\left(-\frac{2\pi}{6} + \frac{\pi}{6}\right)$

$= \frac{1}{2}\left(-\frac{\pi}{6}\right) = -\frac{\pi}{12}$

$\frac{1}{2}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) = \frac{1}{2}\left(\frac{\pi}{6} + \frac{4\pi}{6}\right)$

$= \frac{1}{2}\left(\frac{5\pi}{6}\right)$

$= \frac{5\pi}{12}$

Checkpoint: Lecture 28, problem 4

AMP. $= \left|\frac{3}{2}\right| = \frac{3}{2}$ PERIOD $= \frac{2\pi}{b} = \frac{2\pi}{1/2} = 2\pi\left(\frac{2}{1}\right) = 4\pi$

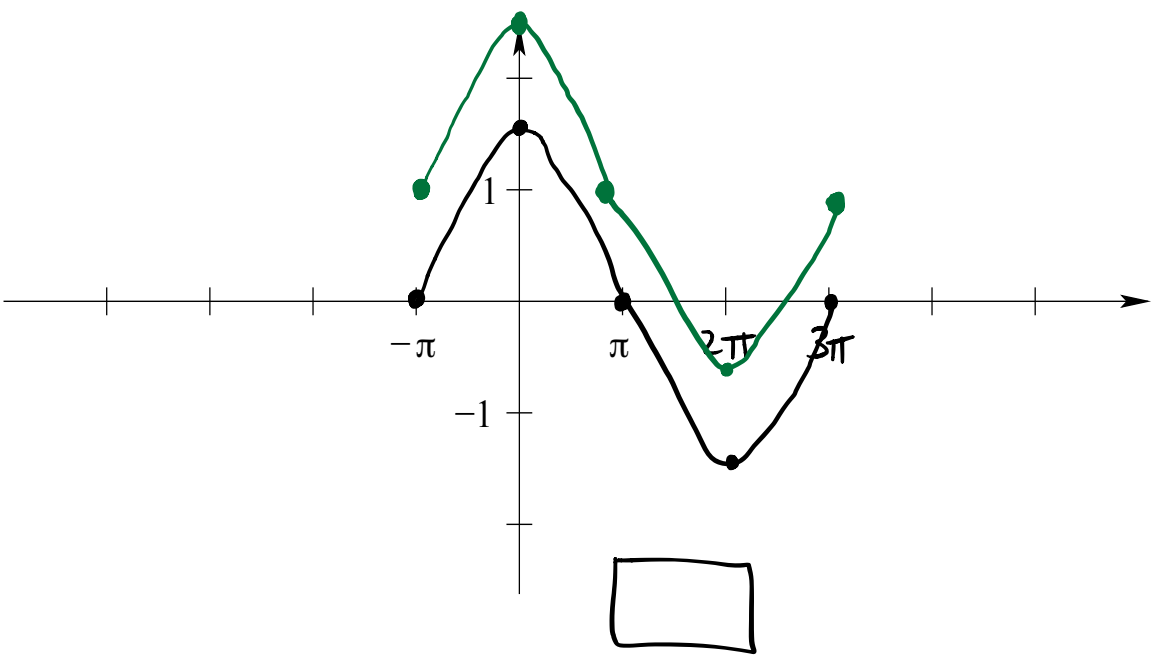
PHASE SHIFT $= \frac{c}{b} = \frac{-\pi/2}{1/2} = -\frac{\pi}{2}\cdot\frac{2}{1} = -\pi$

Vertical Translations

ex. Find the amplitude, period and phase shift of $y = 1 + \frac{3}{2}\sin\left(\frac{x}{2} + \frac{\pi}{2}\right)$, and sketch its graph.

1. GRAPH $y = \frac{3}{2}\sin\left(\frac{x}{2} + \frac{\pi}{2}\right)$

2. SHIFT $y = \frac{3}{2}\sin\left(\frac{x}{2} + \frac{\pi}{2}\right)$ UP 1 TO GET $y = 1 + \frac{3}{2}\sin\left(\frac{x}{2} + \frac{\pi}{2}\right)$



NEW START:

$\frac{x}{2} + \frac{\pi}{2} = 0$

$x = -\pi$

x	$-\pi$	0	π	2π	3π
y	0	$\frac{3}{2}$	0	$-\frac{3}{2}$	0
	0	1	0	-1	0

NEW END:

$\frac{x}{2} + \frac{\pi}{2} = 2\pi$

$\frac{x}{2} = \frac{4\pi}{2} - \frac{\pi}{2} \Rightarrow \frac{x}{2} = \frac{3\pi}{2} \Rightarrow x = 3\pi$

$\frac{1}{2}\left(3\pi + (-\pi)\right) = \frac{1}{2}(2\pi) = \pi$

$\frac{1}{2}(-\pi + \pi) = 0$

ex. As a wave passes by an offshore piling, the height of the water is modeled by the function

$$h(t) = 3\cos\left(\frac{\pi}{10}t\right)$$

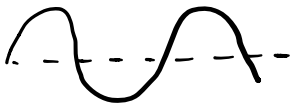
where $h(t)$ is the height in feet above mean sea level at time t seconds.

1) Find the period of the wave.

$$\text{PERIOD} = \frac{2\pi}{\pi/10} = \frac{2\pi}{1} \cdot \frac{10}{\pi} = \boxed{20 \text{ SECONDS}}$$

2) Find the wave height, that is, the vertical distance between the trough and the crest of the wave.

FIND AMPLITUDE AND DOUBLE IT



$$\text{AMP} = |3| = 3$$

$$\Rightarrow 2(3) = 6 \text{ FT.}$$